Rationality-Robust Information Design: Bayesian Persuasion under Quantal Response

Yiding Feng* Chien-Ju Ho† Wei Tang‡

Abstract

Classic mechanism/information design imposes the assumption that agents are *fully rational*, meaning each of them always selects the action that maximizes her expected utility. Yet many empirical evidence suggests that human decisions may deviate from this full rationality assumption. In this work, we attempt to relax the full rationality assumption with *bounded rationality*. Specifically, we formulate the bounded rationality of an agent by adopting the quantal response model (McKelvey and Palfrey, 1995).

We develop a theory of rationality-robust information design in the canonical setting of Bayesian persuasion (Kamenica and Gentzkow, 2011) with binary receiver action. We first identify conditions under which the optimal signaling scheme structure for a fully rational receiver remains optimal or approximately optimal for a boundedly rational receiver. In practice, it might be costly for the designer to estimate the degree of the receiver's bounded rationality level. Motivated by this practical consideration, we then study the existence and construction of *robust* signaling schemes when there is uncertainty about the receiver's bounded rationality level.

1 Introduction

In modern computer science, an important branch of research studies computation that involves multiple parties. One fundamental question raises in this area: how to ensure that different parties do their computation correctly. In the field of mechanism design (which studies protocols for strategic agents), literature imposes the rationality assumption on the participated agents – each agent (a.k.a., party) acts (a.k.a., does computation) in a way to perfectly maximize their own utility. In contrast, the system design literature favors fault tolerant systems (Cristian, 1991; Laprie, 1992; Koren and Krishna, 2020), where the central protocol allows faults within some of the parties. Motivated by the fault tolerance idea from the system design literature, it is interesting to study whether and how the economic lessons derived under the full rationality assumption can be extended to more practical scenarios where agents might make mistakes and thus are *boundedly rational*.

In this paper, we tackle this research direction on relaxing rationality assumption by studying a canonical economic model – persuasion – in information design.[1] In Bayesian persuasion (Kamenica and Gentzkow, 2011), there is a sender and a receiver. Both players have their own utility functions which depend on a state drawn from a common prior as well as an action selected by the receiver. Once the state is realized, the sender observes the realized state, while the receiver only shares a common prior about the state with the sender. The sender can commit to an information structure (a.k.a., signaling scheme) which (possibly randomly) maps the realized state to a signal sent to the receiver. Given the observed signal, the receiver forms a posterior belief about the state and then selects her action (which impacts both her and the sender's utilities). We say the receiver is fully rational if she always (correctly) selects the best action which maximizes her expected utility given her posterior belief about the state.

To capture the possible mistakes which the receiver can make in practice, we relax the full rationality assumption with the bounded rationality modeled by *quantal response* (cf. McKelvey and Palfrey, 1995).[2] To provide informal intuitions, for a fully rational receiver, she takes an action that maximizes her expected utility.

*University of Chicago, yidingfeng@uchicago.edu

†Washington University in St. Louis, chienju.ho@wustl.edu

‡Columbia University, wt2359@columbia.edu

[1] Information design is a field closely related to mechanism design. Mechanism design builds the rule of the game while holding the information structure (i.e., how information is transferred across agents and nature) fixed. In contrast, information design builds information structure while holding the rule of the game fixed.

[2] The quantal response model is also known as multinomial logit model (cf. Talluri and Van Ryzin, 2004) and conditional choice probability (Rust, 1987). There are other models which relax the rationality assumption. See related works for more discussions.

When there is no ties in action utility, this action choice is deterministic. On the other hand, the quantal response accounts for the inherent randomness (and error-proneness) in human decision making and models the human's decision as a probabilistic process. Specifically, in quantal response, for each action the receiver can take, a noise is added into the receiver's utility for taking this action. The receiver then takes an action that maximizes this noisy version of the utility. This noise captures several realistic aspects of human decision making, e.g., when there are additional inherent characteristics in the receiver's utility estimation that we cannot model, or when receiver is drawn from a population and individual differences need to be accounted for. Shifting to the above boundedly rational behavior, two natural questions that our work tries to answer are:

> *Does the structure of optimal signaling scheme for a fully rational receiver preserve or approximately preserve when the receiver is boundedly rational?*
>
> *Can the sender design robust signaling scheme when he has uncertainty of the receiver's boundedly rational behavior?*

To answer the above questions, we focus on Bayesian persuasion with binary receiver action. Though binary receiver action seems a little restrictive at first glance, it is a canonical persuasion model studied extensively in both theoretical computer science and economics literature (see, e.g., Kolotilin et al., 2017; Babichenko and Barman, 2017; Guo and Shmaya, 2019; Xu, 2020; Babichenko et al., 2021). This model, serving as a fundamental cornerstone, has a wide range of applications in practice, including but not limited to product advertising, targeting in sponsored search, recommendation letter, and short video recommendation. See Appendix B.1 for detailed descriptions of these examples. Our results provide both affirmative and negative answers to the above questions, and we underscore that the binary-action setting is sufficiently intricate and challenging enough to establish our main results within our rationality-robust framework. At a high-level, a critical condition influencing our findings is the sender's utility structure, specifically whether it is dependent on the state. [3]

1.1 Main Results and Techniques Based on the practical applications, problems in Bayesian persuasion can be further partitioned into *state independent sender utility (SISU)* environments where the sender's utility does not depend on the realized state; and *state dependent sender utility (SDSU)* environments where the sender's utility depends on both the realized state as well as the receiver's action. For example, as illustrated in Appendix B.1, the aforementioned product advertising and recommendation letter example fall into SISU environments as the seller/advisor only cares whether the buyer buys the product/recruiter hires the student, while short video recommendation and targeting in sponsored search example fall into SDSU environments as the platform's/search engine's revenue also depends on video content/impression attribute.

Revisiting censorship and direct signaling schemes. When the receiver is fully rational, the optimal signaling schemes admit the same structure for both SISU environments and SDSU environments. In a nutshell, the optimal signaling scheme partitions all states into two subsets – high states and low states;[4] and pools all high states into a single signal. On the other side, the signaling structure for low states can be arbitrary and does not affect the optimality of the signaling scheme. Two representative subclasses of signaling schemes have been studied extensively in the literature – *direct signaling schemes* and *censorship signaling schemes*. Both of them pool all high states, but have different signaling structures for low states. Specifically, direct signaling schemes pool all low states, while censorship signaling schemes reveal every low state truthfully. To persuade a fully rational receiver, the sender is indifferent between the optimal direct signaling scheme and the optimal censorship signaling scheme, since both of them maximize the sender's expected utility over all signaling schemes.

The separation of optimal signaling schemes in SISU and SDSU environments. As the first part of our main contributions, for a boundedly rational receiver, we show that in SISU environments, censorship signaling schemes remain optimal, while direct signaling schemes are sub-optimal; and both of them become sub-optimal

[3] As we elaborate later, for a fully rational receiver, the structure of optimal signaling scheme is well-characterized for the binary-action setting, while for the multi-action setting, characterizing a succinct structure that is amenable to theoretical analysis still remains as open question (cf. Dughmi and Xu, 2016; Bergemann et al., 2022a). Thus, studying multi-action setting is beyond the focus on this work, and we leave it as an interesting future direction.

[4] Rigorously speaking, there might exists a threshold state such that a certain fraction of it belongs to high states and the remaining fraction of it belongs to low states.

in SDSU environments. Nonetheless, we also provide the tight approximation bounds of censorship and direct signaling schemes in SDSU environments. Our results (summarized in Table 1) suggest that the structure of optimal signaling schemes for a fully rational receiver is partially preserved (i.e., censorship remains optimal) in SISU environments, and approximately preserved (i.e., up to an $\Theta(m)$-approximation factor) in SDSU environments, where m denotes the number of the states. Moreover, to persuade a boundedly rational receiver, the sender prefers censorship than direct signaling schemes.

Table 1: Approximation ratio of censorship/direct signaling schemes under bounded rationality. The number of states is m.

	censorship signaling schemes	direct signaling schemes
SISU	1 [Theorem 3.1]	$\Theta(m)$ [Theorem 3.2, Theorem 4.2]
SDSU	$\Theta(m)$ [Proposition 4.1, Theorem 4.2]	

In more detail, in SISU environments, we show that for any boundedly rational receiver, censorship signaling scheme is optimal among all signaling schemes (Theorem 3.1) and direct signaling scheme is $\Omega(m)$-approximation where m is the number of states (Theorem 3.2). We further provide structural characterizations on how to determine the high/low states partition in the optimal censorship signaling scheme. In particular, for a receiver with any bounded rationality level, including a fully rational receiver, the subset of high states is *nested* (i.e., increasing) with respect to the bounded rationality level. Namely, the optimal signaling scheme reveals less information for a more rational receiver.

To show the optimality of censorship signaling schemes for a boundedly rational receiver in SISU environments, we first introduce a linear program $\mathcal{P}_{\texttt{OPT-Primal}}$, in which the constraints regulate the set of all feasible signaling schemes, and the objective function computes the expected sender utility of a given signaling scheme. This linear program $\mathcal{P}_{\texttt{OPT-Primal}}$ is inspired by a connection between our problem and public Bayesian persuasion for a continuum population of fully rational receivers with a specific utility structure. Given the linear program $\mathcal{P}_{\texttt{OPT-Primal}}$ and its dual program, we characterize the censorship structure in the optimal signaling scheme by constructing a dual assignment explicitly and then invoke the strong duality of linear programs.

In SDSU environments, the optimal signaling scheme no longer admits the censorship nor direct structure. We start by providing a SDSU example (Example 4.1) and showing that the approximation of every censorship (resp. direct) signaling scheme is $\Omega(m)$ (Proposition 4.1). En route to proving this lower bound, we present a stronger result, namely that any signaling scheme must be an $\Omega(m/L)$-approximation where L is the maximum number of signals induced by a state in this signaling scheme (Theorem 4.1). Next, we provide the matching upper bound that for any problem instance with m states, there exists a censorship (resp. direct) signaling scheme that is an $O(m)$-approximation to the optimal signaling scheme (Theorem 4.2).

The key step in establishing the $O(m)$-approximation upper bounds for censorship (resp. direct) signaling schemes (Theorem 4.2) is that we characterize a 4-approximation signaling scheme that uses $O(m)$ signals and has the following two structural properties (Lemma 4.3): (i) every signal is used to pool at most two states; (ii) every pair of states is pooled at most one signal. Intuitively, property (i) says that, in the signaling scheme that we characterize, a signal either fully reveals the state or randomizes receiver's uncertainty only on two states, and property (ii) says that there is no need for the sender to pool a pair of states at multiple signals in order to have 4-approximation. We then leverage the structure of this 4-approximation signaling scheme to show the existence of $O(m)$-approximation censorship (resp. direct) signaling schemes. To prove this technical lemma (Lemma 4.3), we build a connection between the signaling schemes satisfying properties (i) (ii) with fractional solutions in the *generalized assignment problem* (Shmoys and Tardos, 1993). In particular, focusing on signaling schemes that have properties (i) (ii), we introduce a linear program which shares the same format as the linear program relaxation of the generalized assignment problem. For generalized assignment problem, Shmoys and Tardos (1993) show that the optimal integral solution is a 2-approximation to the optimal fractional solution. We argue that the optimal integral solution can be converted into a feasible signaling scheme that has properties (i) (ii), uses $O(m)$ signals, and suffers an additional two factor loss in its payoff.

Rationality-robust information design. As the second part of our main contributions, we introduce *rationality-*

robust information design – a framework in which a signaling scheme is designed for a receiver whose bounded rationality level is unknown. In our previous discussions, designing optimal signaling schemes in both SISU and SDSU environments is rationality-oriented – the sender needs to know exactly the receiver's bounded rationality level. In practice, the sender may not be able to have (or require significant cost to learn) such perfect knowledge. Motivated by this concern, the goal of rationality-robust information design is to identify robust signaling schemes – ones with good (multiplicative) approximation to the optimal signaling scheme that is tailored to any possible bounded rationality level of the receiver. Similar to our results above, we observe that obtaining rationality-robust signaling scheme is much more tractable in SISU environments than SDSU environments.

In SISU environments, by leveraging the structural property we mentioned before (i.e., the optimal (censorship) signaling scheme reveals less information for a more rational receiver), we show that the optimal censorship for a fully rational receiver achieves a 2 rationality-robust approximation when the sender has no knowledge of the receiver's bounded rationality level (Theorem 5.1). We also provide an example to show the tightness of the result (Proposition 5.1). Our result suggests that, *up to a two factor, the knowledge of the receiver's bounded rationality level are unimportant in SISU environments.* For the comparison, we also show that the optimal direct signaling scheme for a fully rational receiver achieves unbounded rationality-robust approximation (Proposition 5.2). This repeats the takeaway mentioned above – the sender prefers censorship than direct signaling schemes in SISU environments under bounded rationality.

In contrast, in SDSU environments, we show that there exists no signaling scheme with bounded rationality-robust approximation ratio, when the sender has no knowledge of the receiver's bounded rationality level (Theorem 5.2), and this result holds even if the state space is binary. Our result suggests that, *there exists a tradeoff between the knowledge of the receiver's rationality level and the achievable rationality-robustness in SDSU environments.* To show this impossibility result, we construct a binary-state problem instance and a set of carefully chosen possible bounded rationality levels. The key to our approach is by introducing a factor-revealing program to lower bound the optimal rationality-robust approximation ratio. By analyzing its dual program, we show that the rationality-robust approximation ratio of any signaling scheme is unbounded. This impossibility result indicates that there exists a tradeoff between the knowledge of the receiver's rationality level and the achievable rationality-robustness. Though it appears challenging to obtain a bounded-factor rationality-robust approximation for arbitrary set of rationality levels, and general problem instances in SDSU environments, we obtain a preliminary positive result under a boundedness condition on the receiver's rationality level. In particular, when the state space is binary, under a reasonable multiplicative boundedness condition (i.e., learning the receiver's bounded rationality level up to a multiplicative error) on the receiver's bounded rationality levels, we show that the sender is able to design a signaling scheme whose rationality-robust approximation ratio depends linearly on the multiplicative error (Proposition F.1).

1.2 Related Work In this section, we discuss the works that are closely related to our work, and we discuss further related work in Appendix A.

There has been a growing interest in understanding how to design robust signaling schemes in the face of uncertain receiver behavior. Our work contributes to this line of research by studying robust signaling schemes when the receiver's bounded rationality level is unknown. The approach we take is similar to the approach often used in prior-independent mechanism design, examining the *approximation ratios* of the designed mechanisms. Our work differs from previous works that either focus on the regret minimization Babichenko et al. (2021); Chen and Lin (2023) or minimax approach Dworczak and Pavan (2022); Kosterina (2022); Hu and Weng (2021). Notably, Babichenko et al. (2021) present a negative result saying that there exists no nontrivial bound of the additive regret if the sender has no knowledge about the receiver's utilities. While this result shares a similar message to our impossibility result in Section 5, there are notable differences between the two studies that preclude direct comparison: (i) our impossibility result is under SDSU setting, whereas theirs is under SISU setting (for which we have a positive result); (ii) the adversary in our setting is limited to choosing the receiver's behavior (i.e., the rationality level) in the quantal response model, whereas theirs considers a worst-case adversary. Recent work by Chen and Lin (2023) also examines the design of robust signaling schemes for non-best-responding receiver, but with a focus on the regret minimization approach. Our persuasion setting with fully rational receiver can also be viewed as a Stackelberg game where the sender moves first by committing to a signaling scheme, and the receiver takes an action that best responds to sender's signaling scheme. With bounded rationality, receiver in our setting is not best-responding to sender's signaling scheme. This shares similarity to recent work by Gan et al.

(2023) who study Stackelberg games with suboptimal follower response. However, their work adopts a worst-case perspective and considers the worst possible follower behavior up to some plausible ranges, while our work adopts a model-based approach and the follower is responding with following a quantal response model.

2 Preliminaries

2.1 Model and Problem Definition In this paper, we study the persuasion problem for a receiver with bounded rationality. There are two players, a sender and a receiver. There is an unknown state θ drawn from a finite set $[m] \triangleq \{1, 2, \dots, m\}$ according to a prior distribution $\lambda \in \Delta([m])$, which is common knowledge among both players. Throughout the paper, we use θ to denote the state as a random variable, and $i, j, k \in [m]$ as its possible realization. We use λ_i to denote the probability that the realized state is $i \in [m]$, i.e., $\lambda_i \triangleq \Pr[\theta = i]$. The receiver has a binary action set $\mathcal{A} = \{0, 1\}$. Given a realized state $i \in [m]$, by taking action $a \in \mathcal{A}$, the utility of the receiver is $v_i(a)$ and the utility of the sender is $u_i(a)$. Following the standard convention (e.g., Anunrojwong et al., 2020; Alonso and Câmara, 2016b; Babichenko et al., 2021; Lingenbrink and Iyer, 2019), Throughout this paper, we focus on the setting where $u_i(1) \ge u_i(0)$ for all $i \in [m]$, and normalize $u_i(0) \equiv 0$ and denote $u_i \triangleq u_i(1)$. [5]

The objective of the sender is to maximize his expected utility. Before the state θ is realized, the sender commits to a signal space Σ and a signaling scheme $\pi : [m] \to \Delta(\Sigma)$, a mapping from the realized state into probability distributions over signals. We use $\pi_i(\sigma)$ to denote the probability that signal $\sigma \in \Sigma$ is realized when the realized state is state i. Upon seeing signal σ, the receiver performs a Bayesian update and infers a posterior belief over the state. In particular, the posterior probability of state i given realized signal σ is $\mu_i(\sigma) \triangleq \frac{\lambda_i \pi_i(\sigma)}{\sum_{j\in[m]} \lambda_j \pi_j(\sigma)}$.

In this paper, we assume that the receiver is boundedly rational by modeling her as a (logit) quantal response player (McKelvey and Palfrey, 1995). Specifically, instead of taking the best response which maximizes the expected utility, a quantal player randomly selects an action with probability proportional to the expected utility. In our model, given posterior belief $\mu \in \Delta([m])$ and its induced expected utility $v(a \mid \mu) \triangleq \sum_{i\in[m]} \mu_i \, v_i(a)$ for each action $a \in \mathcal{A}$, the receiver selects action 1 with probability[6]

$$\frac{\exp(\beta \cdot v(1 \mid \mu))}{\exp(\beta \cdot v(1 \mid \mu)) + \exp(\beta \cdot v(0 \mid \mu))} = \frac{1}{1 + \exp(\beta \cdot (v(0 \mid \mu) - v(1 \mid \mu)))}$$

Here $\beta \in [0, \infty)$ is the bounded rationality level. When the bounded rationality level β equals zero, the receiver takes each action uniformly at random regardless of her posterior belief. When the bounded rationality level β equals infinite, our model recovers the classic Bayesian persuasion for a (fully) rational receiver who takes the action which maximizes her expected utility.

Let function[7] $W^{(\beta)}(x) \triangleq 1/(1 + \exp(\beta x))$. When the bounded rationality level β is clear from the context, we simplify $W^{(\beta)}$ with W. Given any posterior belief μ, we have $v(0 \mid \mu) - v(1 \mid \mu) = \sum_{i\in[m]} \mu_i \, v_i$, where $v_i \triangleq v_i(0) - v_i(1)$ represents how much the receiver prefers action 0 over action 1 given state i. Without loss of generality, we assume $\{v_i\}$ is strictly increasing in i. With the above definitions, we can rewrite the probability that the receiver takes action 1 as $W\left(\sum_{i\in[m]} \mu_i \, v_i\right)$. Intuitively speaking, since the probability that receiver takes action 1 only depends on the expected utility difference $\sum_{i\in[m]} \mu_i \, v_i$, it is without loss of generality to restrict to signaling scheme where each signal δ represents its induced expected utility difference, i.e., $\delta \equiv \sum_{i\in[m]} \mu_i(\delta) \, v_i$. Recall that $\mu_i(\delta)$ is the posterior probability of state i given realized signal δ. We formalize this idea by writing our problem as the following linear program $\mathcal{P}_{\texttt{OPT-Primal}}$ (and its dual program $\mathcal{P}_{\texttt{OPT-Dual}}$) with variables $\{\pi_i(\delta)\}_{\delta\in\mathbb{R}, i\in[m]}$.[8] See

[5] A discussion of how our results could be extended without this assumption is provided in Appendix B.2.

[6] One explanation of this quantal response behavior is that the receiver faces a random shock when she is making the decision. See Appendix B.4 for more details.

[7] We note that many results in Section 3, Section 4 hold for general function W. See Appendix B.3 for detailed discussions.

[8] While we allow signaling schemes to have continuous signal space Σ, i.e., $\pi_i(\cdot)$ can be interpreted as a probability density function over Σ, all signaling schemes (as well as the optimal signaling schemes) considered in this paper have finite signal space. Therefore, we abuse the notation and use $\pi_i(\cdot)$ as the probability mass function when it is clear from the context.

Proposition 2.1 and its proof in Appendix C.

$$(\mathcal{P}_{\texttt{OPT-Primal}}) \qquad \begin{array}{lllll} \max\limits_{\boldsymbol{\pi}\geq \mathbf{0}} & \sum_{i\in[m]} \lambda_i u_i \int_{-\infty}^{\infty} \pi_i(\delta) W(\delta)\, d\delta & \text{s.t.} & \\ & \sum_{i\in[m]} \lambda_i (v_i - \delta)\, \pi_i(\delta) = 0 & \delta \in (-\infty,\infty) & \langle \alpha(\delta)\rangle \\ & \int_{-\infty}^{\infty} \pi_i(\delta) d\delta = 1 & i\in[m] & \langle \eta(i)\rangle \end{array}$$

$$(\mathcal{P}_{\texttt{OPT-Dual}}) \qquad \begin{array}{llll} \min\limits_{\boldsymbol{\alpha},\boldsymbol{\eta}} & \sum_{i\in[m]} \eta(i) & \text{s.t.} & \\ & \lambda_i (v_i - \delta)\, \alpha(\delta) + \eta(i) \geq \lambda_i u_i W(\delta) & \delta\in(-\infty,\infty), i \in [m] & \langle \pi_i(\delta)\rangle \end{array}$$

Proposition 2.1. *For every feasible solution $\{\pi_i(\delta)\}$ in program $\mathcal{P}_{\texttt{OPT-Primal}}$, there exists a signaling scheme where for each state $i \in [m]$, the boundedly rational receiver takes action 1 with probability $\int_{-\infty}^{\infty} \pi_i(\delta) W(\delta)\, d\delta$. Furthermore, the sender's optimal expected utility (in the optimal signaling scheme) is equal to the optimal objective value of program $\mathcal{P}_{\texttt{OPT-Primal}}$.*

The first constraint in the program $\mathcal{P}_{\texttt{OPT-Primal}}$ ensures that whenever a signal δ is realized, the probability for the receiver for taking action 1 is exactly $W(\delta)$. Due to Proposition 2.1, in the remaining of the paper, we describe signaling schemes by $\{\pi_i(\delta)\}$ as the feasible solutions in program $\mathcal{P}_{\texttt{OPT-Primal}}$, and $\{\pi_i(\sigma)\}$ as the original definition interchangeably. We use $\mathbf{Payoff}_\beta[\pi]$ to denote the expected sender utility of signaling scheme π (i.e., the objective value for feasible solution π in program $\mathcal{P}_{\texttt{OPT-Primal}}$) for a receiver with bounded rationality level β. We drop subscript β in $\mathbf{Payoff}_\beta[\cdot]$ when it is clear from the context.

Our persuasion problem for the boundedly rationally receiver is equivalent to a public persuasion problem for a continuum population of rational receivers with a specific utility structure, and thus program $\mathcal{P}_{\texttt{OPT-Primal}}$ can be reinterpreted as the program for this public persuasion problem. See Appendix B.4 for more details.

2.2 Optimal Signaling Schemes for A Fully Rational Receiver Here we introduce two subclasses of signaling schemes – *censorship signaling schemes* and *direct signaling schemes*, that will be discussed throughout this paper. Briefly speaking, a censorship (resp. direct) signaling scheme partitions the state space $[m]$ into three disjoint subsets:[9] high states $\mathcal{H}$, threshold state $\{i^\dagger\}$, and low states $\mathcal{L}$, and specifies a threshold state probability $p^\dagger \in [0,1]$. It pools all states in $\mathcal{H}$ as well as a $(p^\dagger)$-fraction of the threshold state $i^\dagger$ into a pooling signal $\delta^\dagger$, and fully reveals other states (resp. pools all other states into another pooling signal $\delta^\ddagger$). See Definition 2.1 and Definition 2.2 for the formal definitions.[10]

Definition 2.1. *A* censorship signaling scheme*, parameterized by a state space partition $\mathcal{H} \sqcup \{i^\dagger\} \sqcup \mathcal{L}$, and threshold state probability $p^\dagger$ admits the form as follows*

$$\begin{aligned} i \in \mathcal{H}: \qquad & \pi_i(\delta) = \mathbb{1}\left[\delta = \delta^\dagger\right] \\ & \pi_{i^\dagger}(\delta) = p^\dagger \cdot \mathbb{1}\left[\delta = \delta^\dagger\right] + (1-p^\dagger)\cdot \mathbb{1}[\delta = v_{i^\dagger}] \\ i \in \mathcal{L}: \qquad & \pi_i(\delta) = \mathbb{1}[\delta = v_i] \end{aligned}$$

where $\delta^\dagger = \frac{p^\dagger \lambda_{i^\dagger} v_{i^\dagger} + \sum_{i\in\mathcal{H}} \lambda_i v_i}{p^\dagger \lambda_{i^\dagger} + \sum_{i\in\mathcal{H}} \lambda_i}$ is the pooling signal.

Definition 2.2. *A* direct signaling scheme*, parameterized by a state space partition $\mathcal{H} \sqcup \{i^\dagger\} \sqcup \mathcal{L}$, and threshold state probability $p^\dagger$ admits the form as follows*

$$\begin{aligned} i \in \mathcal{H}: \qquad & \pi_i(\delta) = \mathbb{1}\left[\delta = \delta^\dagger\right] \\ & \pi_{i^\dagger}(\delta) = p^\dagger \cdot \mathbb{1}\left[\delta = \delta^\dagger\right] + (1-p^\dagger)\cdot \mathbb{1}\left[\delta = \delta^\ddagger\right] \\ i \in \mathcal{L}: \qquad & \pi_i(\delta) = \mathbb{1}\left[\delta = \delta^\ddagger\right] \end{aligned}$$

[9] Namely, $\mathcal{H} \cup \{i^\dagger\} \cup \mathcal{L} = [m]$, $\mathcal{H} \cap \mathcal{L} = \emptyset$, $i^\dagger \notin \mathcal{H}$, and $i^\dagger \notin \mathcal{L}$.

[10] Our definition is equivalent to censorship signaling schemes for persuasion problem with continuous state space (see, e.g., Dworczak and Martini, 2019) by considering the quantile space of state space $[m]$.

where $\delta^\dagger = \frac{p^\dagger \lambda_{i^\dagger} v_{i^\dagger} + \sum_{i\in\mathcal{H}} \lambda_i v_i}{p^\dagger \lambda_{i^\dagger} + \sum_{i\in\mathcal{H}} \lambda_i}$ *and* $\delta^\ddagger = \frac{(1-p^\dagger)\lambda_{i^\dagger} v_{i^\dagger} + \sum_{i\in\mathcal{L}} \lambda_i v_i}{(1-p^\dagger)\lambda_{i^\dagger} + \sum_{i\in\mathcal{L}} \lambda_i}$ *are the pooling signals.*

The only difference between censorship signaling schemes and direct signaling schemes is the signaling structure for the $(1-p^\dagger)$-fraction of the threshold state $i^\dagger$ and every state in $\mathcal{L}$ – censorship signaling schemes fully reveal them, while direct signaling schemes pools them all together. As a sanity check, note that censorship (resp. direct) signaling schemes are indeed the feasible solutions of program $\mathcal{P}_{\texttt{OPT-Primal}}$. We also highlight two standard censorship signaling schemes: the *full-information revealing signaling scheme* which reveals all states separately, and the *no-information revealing signaling scheme* which pools all states at a single signal.

When the receiver is fully rational (i.e., the bounded rationality level $\beta = \infty$), there exists a censorship (resp. direct) signaling scheme that is indeed optimal.

LEMMA 2.1. *[See for example Renault et al., 2017] For a fully rational receiver (i.e., with bounded rationality level $\beta = \infty$), it is optimal for the sender to adopt a censorship (resp. direct) signaling scheme such that*

(i) threshold state $i^\dagger = \arg\max_{i\in[m]} \left\{ \frac{v_i}{u_i} : \sum_{j:\frac{v_j}{u_j}<\frac{v_i}{u_i}} \lambda_j v_j \le 0 \right\}$;

(ii) high states $\mathcal{H} = \left\{ i \in [m] : \frac{v_i}{u_i} < \frac{v_{i^\dagger}}{u_{i^\dagger}} \right\}$, *and low states* $\mathcal{L} = \left\{ i \in [m] : \frac{v_i}{u_i} > \frac{v_{i^\dagger}}{u_{i^\dagger}} \right\}$;

(iii) threshold state probability $p^\dagger = \max \left\{ p \in [0,1] : p\lambda_{i^\dagger} v_{i^\dagger} + \sum_{i\in\mathcal{H}} \lambda_i v_i \le 0 \right\}$.

In fact, to achieve the optimality for a fully rational receiver, it only requires that all states in $\mathcal{H}$ together with $(p^\dagger)$-fraction of threshold state $i^\dagger$ are pooled into signal $\delta^\dagger$ where the assignments of $\mathcal{H}, i^\dagger, p^\dagger$ and $\delta^\dagger$ are defined in Lemma 2.1 (Renault et al., 2017). In other words, no restrictions on the signaling structure on the remaining $(1-p^\dagger)$-fraction of threshold state $i^\dagger$ and other states in $\mathcal{L}$ are required. In this sense, the optimal censorship and optimal direct signaling scheme can be thought as two extreme cases on the signaling structure for those states, i.e., fully revealing them or pooling them all together, and both achieve the optimality over all signaling schemes. Therefore, for a fully rational receiver, the sender is indifferent between the optimal censorship and the optimal direct signaling schemes. However, as we shown in the later sections, there exists a separation between these two types of signaling schemes when the receiver is boundedly rational.

3 State Independent Sender Utility (SISU) Environments

In this section, we consider the *state independent sender utility (SISU)* environments where the sender's utility $\{u_i\}_{i\in[m]}$ is independent of the realized state. Namely, we assume $u_i \equiv 1$ for every state $i \in [m]$. Furthermore, for ease of presentation, this section assumes $v_1 < 0$ and $v_m > 0$.[11]

Recall that for a fully rational receiver, Lemma 2.1 shows the optimality of both censorship signaling schemes and direct signaling schemes. However, when the receiver is boundedly rational, there exists a separation between these two subclasses of signaling schemes. As the main result of this section, Theorem 3.1 in Section 3.1 shows that in SISU environments, for a boundedly rational receiver, it is optimal for the sender to adopt a censorship signaling scheme. In contrast, Theorem 3.2 in Section 3.2 shows that there exists a SISU problem instance, where any direct signaling scheme is $\Omega(m)$-approximation.

3.1 Censorship as Optimal Signaling Schemes In this subsection, we show that in SISU environments, for a receiver with bounded rationality level β, it is optimal for the sender to adopt a censorship signaling scheme. Our result recovers the optimal censorship signaling scheme of a fully rational receiver (Lemma 2.1). In other words, the optimality of the censorship signaling schemes is preserved even when the receiver is boundedly rational in SISU environments.

THEOREM 3.1. *In SISU environments, for a boundedly rational receiver with any bounded rationality level β, there exists a censorship signaling scheme π^* that is the optimal signaling scheme. Specifically,*

[11] If $v_i \le 0, \forall i \in [m]$, we can introduce one dummy state $m+1$ such that $v_{m+1} = 1$ and $\lambda_{m+1} = 0$. Similarly, we can add one dummy state if $v_i \ge 0, \forall i \in [m]$. Thus, $v_1 < 0$ and $v_m > 0$ is without loss of generality.

(i) the threshold state $i^\dagger$ and the threshold state probability $p^\dagger$, together with an auxiliary variable $\delta^\ddagger$, solve the following feasibility program $\mathcal{P}_{\texttt{SISU-OPT}}$:

$$
(\mathcal{P}_{\texttt{SISU-OPT}}) \qquad
\begin{aligned}
(1-p^\dagger)(\delta^\ddagger - v_{i^\dagger}) &= 0 && \textit{(complementary-slackness)} \\
\big(W(\delta^\ddagger) - W(\delta^\dagger)\big) - W'(\delta^\dagger)(\delta^\ddagger - \delta^\dagger) &= 0 && \textit{(dual-feasibility-1)} \\
\max_{i\in[m]}\{v_i : v_i \le \delta^\ddagger\} &= v_{i^\dagger} && \textit{(dual-feasibility-2)} \\
\delta^\dagger \le 0,\ \delta^\ddagger &\ge 0 && \textit{(dual-feasibility-3)} \\
0 \le p^\dagger &\le 1 && \textit{(primal-feasibility)}
\end{aligned}
$$

where $\delta^\dagger = \frac{\sum_{i:i<i^\dagger}\lambda_i v_i + p^\dagger \lambda_{i^\dagger} v_{i^\dagger}}{\sum_{i:i<i^\dagger}\lambda_i + p^\dagger\lambda_{i^\dagger}}$ is the pooling signal;

(ii) high states $\mathcal{H} = \{i \in [m] : i < i^\dagger\}$, and low states $\mathcal{L} = \{i \in [m] : i > i^\dagger\}$.

In below, we first provide intuitions behind the constraints in the feasibility program $\mathcal{P}_{\texttt{SISU-OPT}}$, and the properties as well as the implications of the above characterized optimal censorship signaling scheme. Then, we provide the high-level proof idea of Theorem 3.1.

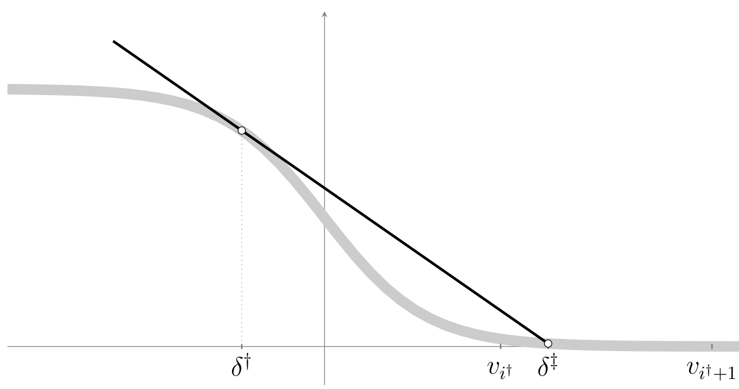


Figure 1: Graphical illustration for constraints `dual-feasibility-1`, `dual-feasibility-2` and `dual-feasibility-3` in feasibility program $\mathcal{P}_{\texttt{SISU-OPT}}$. The gray solid curve is function $W(\cdot)$. Fix an arbitrary $\delta^\ddagger \in [0,\infty)$. Constraint `dual-feasibility-1` uniquely pins down $\delta^\dagger \in (-\infty, 0]$ (and thus constraint `dual-feasibility-3` is satisfied as well) such that the black solid line through point $(\delta^\dagger, W(\delta^\dagger))$ and point $(\delta^\ddagger, W(\delta^\ddagger))$ is tangent to curve $W(\cdot)$ at point $(\delta^\dagger, W(\delta^\dagger))$. Constraint `dual-feasibility-2` uniquely pins down $i^\dagger \triangleq \arg\max_i\{v_i : v_i \le \delta^\ddagger\}$. The tangent line and the curve $W(\delta), \forall \delta \ge \delta^\ddagger$ forms a upper convex envelop for function W.

Graphical interpretation of optimal signaling scheme. To develop intuition for optimal censorship, we start with constraints `dual-feasibility-1` and `dual-feasibility-3`. Recall that $W(x) = 1/(1+\exp(\beta x))$ is concave in $(-\infty, 0]$ and convex in $[0,\infty)$. Constraint `dual-feasibility-1` has the following graphical interpretation: the line through point $(\delta^\dagger, W(\delta^\dagger))$ and point $(\delta^\ddagger, W(\delta^\ddagger))$ is tangent to curve $W(\cdot)$ at point $(\delta^\dagger, W(\delta^\dagger))$. Notably, for every $\delta^\ddagger \ge 0$, there exists a unique $\delta^\dagger \le 0$ which satisfies `dual-feasibility-1`. In particular, the mapping from $\delta^\ddagger \in [0,\infty)$ to $\delta^\dagger \in (-\infty, 0]$ satisfying `dual-feasibility-1` is monotone decreasing and is a bijection (See Figure 1 for illustration). Constraint `dual-feasibility-2` means that threshold state $i^\dagger$ is the largest state index such that $v_i \le \delta^\ddagger$, i.e., $i^\dagger = \arg\max_i\{v_i : v_i \le \delta^\ddagger\}$.[12] Hence, starting with an arbitrary $\delta^\ddagger \ge 0$, constraints `dual-feasibility-1` and `dual-feasibility-2` pin down a unique tuple $(\delta^\dagger, \delta^\ddagger, i^\dagger, p^\dagger)$: constraint `dual-feasibility-1` pins down a unique $\delta^\dagger \le 0$, constraint `dual-feasibility-2` pins down a unique $i^\dagger$, and then $p^\dagger$ is uniquely determined as well by the relation between $\delta^\dagger, i^\dagger, p^\dagger$.

Essentially, the tangent line segment from point $(\delta^\dagger, W(\delta^\dagger))$ to point $(\delta^\ddagger, W(\delta^\ddagger))$ and the part of curve $W(\delta), \forall\delta \ge \delta^\ddagger$ form a *upper convex envelop* for function $W(\cdot)$. It is easy to see that there exist infinitely many such

[12]Recall we assume $v_1 < 0$ without loss of generality, and thus $i^\dagger = \arg\max_i\{v_i : v_i \le \delta^\ddagger\}$ is well-defined for $\delta^\ddagger \ge 0$.

upper convex envelops for function $W(\cdot)$. However, the optimal censorship is the unique one that the corresponding envelop ensures `primal-feasibility` and satisfies `complementary-slackness` for the assignment on the threshold state $i^\dagger$.

Less rational, more information revealing. In SISU environments, for both fully rational receiver and boundedly rational receiver, it is optimal for the sender to adopt censorship signaling schemes (Lemma 2.1, Theorem 3.1).[13] [14] However, in the optimal censorship for different rationality levels, the threshold state $i^\dagger$ and the threshold state probability $p^\dagger$ may not be the same. For example, consider an instance where $\sum_{i\in[m]} \lambda_i v_i < 0$. Lemma 2.1 suggests that the optimal censorship $\hat{\pi}^*$ for a fully rational receiver selects the threshold state $\hat{i}^\dagger$ and threshold state probability $\hat{p}^\dagger$ such that the pooling signal $\hat{\delta}^\dagger = 0$. In contrast, Theorem 3.1 suggests that the optimal censorship π^* selects the threshold state $i^\dagger$ and threshold state probability $p^\dagger$ such that the pooling signal $\delta^\dagger \le 0$. Thus, in this instance, the optimal censorship $\hat{\pi}^*$ for a fully rational receiver pools more states than the optimal censorship π^* for a boundedly rational receiver, i.e., $\hat{\mathcal{H}} \supseteq \mathcal{H}$. Here we generalize this observation and show the monotonicity of threshold state $i^\dagger$ and threshold state probability $p^\dagger$ with respect to the rationality level β. Its proof is based on the analysis for the feasibility program $\mathcal{P}_{\texttt{SISU-OPT}}$, which we defer to Appendix D.2.

PROPOSITION 3.1. *In SISU environments, let π^* (resp. $\hat{\pi}^*$) be the optimal censorship for a boundedly rational receiver with boundedly rational level β (resp. $\hat{\beta}$). If $\beta \le \hat{\beta}$, then the threshold state $i^\dagger$ in π^* is weakly smaller than the threshold state $\hat{i}^\dagger$ in $\hat{\pi}^*$, i.e., $i^\dagger \le \hat{i}^\dagger$; and threshold state probability $p^\dagger \le \hat{p}^\dagger$.*

One concrete insight behind the above result is that: The optimal (censorship) signaling scheme requires the sender to reveal more information (i.e., Blackwell ordering, Blackwell, 1953) for a less rational receiver. This insight can be also developed from the curvature of the function $W(\cdot)$. When the receiver is less rational, i.e., the rationality level β becomes smaller, the curve $W(\cdot)$ becomes flatter. Hence, the tangent point $(\delta^\dagger, W(\delta^\dagger))$ is farther away from the point $(\delta^\ddagger, W(\delta^\ddagger))$, and the pooling probability $p^\dagger$ has to be smaller to make the pooling signal $\delta^\dagger$ relatively smaller. Thus, the threshold state and the threshold state probability must decrease in order to satisfy the feasibility constraint in the program $\mathcal{P}_{\texttt{SISU-OPT}}$, which leads to more information revealing.

Proof overview of Theorem 3.1. Now we first provide a proof overview for Theorem 3.1, and in the sequel, we present the detailed proof. At the heart of proof of Theorem 3.1, we use the strong duality between the primal program $\mathcal{P}_{\texttt{OPT-Primal}}$ and its dual program $\mathcal{P}_{\texttt{OPT-Dual}}$. Specifically, given a feasible solution in feasibility program $\mathcal{P}_{\texttt{SISU-OPT}}$, we explicitly construct a feasible primal assignment in $\mathcal{P}_{\texttt{OPT-Primal}}$ and a feasible dual assignment in $\mathcal{P}_{\texttt{OPT-Dual}}$ and show the complementary slackness holds.[15] In the formal proof, for each possible tuple $(\delta^\dagger, \delta^\ddagger, i^\dagger, p^\dagger)$ described in the graphical interpretation (i.e., satisfying constraints `dual-feasibility-1`, `dual-feasibility-2`, `dual-feasibility-3`), we can construct a feasible assignment for dual program $\mathcal{P}_{\texttt{OPT-Dual}}$. Notably, each feasible solution to the dual program $\mathcal{P}_{\texttt{OPT-Dual}}$ forms a upper convex envelop for the function $W(\cdot)$. To finish the proof with strong duality, we require such tuple to additionally satisfy constraint `primal-feasibility` to ensure the feasibility of the constructed primal assignment, and constraint `complementary-slackness` to ensure the complementary slackness of the constructed assignment on the threshold state $i^\dagger$. The existence and uniqueness of such tuple is shown in Lemma 3.1, its proof is in Appendix D.3.

LEMMA 3.1. *There exists a unique feasible solution in program $\mathcal{P}_{\texttt{SISU-OPT}}$.*

Proof of Theorem 3.1. We prove the optimality of the signaling scheme π^* defined in Theorem 3.1 by constructing a feasible dual solution to the dual program $\mathcal{P}_{\texttt{OPT-Dual}}$ that satisfies the complementary slackness. Let $(\delta^\dagger, \delta^\ddagger, i^\dagger, p^\dagger)$ be the unique feasible solution to program $\mathcal{P}_{\texttt{SISU-OPT}}$.

[13] Recall we assume $\{v_i\}$ is weakly increasing and $u_i \equiv 1$ for every state $i \in [m]$. Thus, $i < i^\dagger$ if and only if $v_i/u_i \le v_{i^\dagger}/u_{i^\dagger}$ and the construction of optimal censorship in Theorem 3.1 recovers the construction in Lemma 2.1 for a fully rational receiver in the SISU environments.

[14] The feasibility program $\mathcal{P}_{\texttt{SISU-OPT}}$ also recovers the structure of the optimal censorship for a fully rational receiver in SISU environments. See Appendix D.1 for detailed discussions.

[15] Here the name of each constraint in $\mathcal{P}_{\texttt{SISU-OPT}}$ indicates its usage in the assignment construction.

Assignment construction. To facilitate the analysis, we explicitly write out the optimal signaling scheme π^* as follows,

$$\begin{aligned} i \in [i^\dagger - 1]: \quad & \pi_i^*(\delta^\dagger) = 1; \\ & \pi_{i^\dagger}^*(\delta^\dagger) = p^\dagger, \qquad \pi_{i^\dagger}^*(v_{i^\dagger}) = 1 - p^\dagger; \\ i \in [i^\dagger + 1 : m]: \quad & \pi_i^*(v_i) = 1 \end{aligned}$$

Due to constraint `primal-feasibility` in program $\mathcal{P}_{\texttt{SISU-OPT}}$, signaling scheme π^* is feasible. Now, consider the following dual assignment $\{\alpha(\delta), \eta(i)\}$ of program $\mathcal{P}_{\texttt{OPT-Dual}}$,

$$\begin{aligned} \delta \in (\infty, \delta^\ddagger]: \quad & \alpha(\delta) = -W'(\delta^\dagger) \\ \delta \in [\delta^\ddagger, v_{i^\dagger+1}]: \quad & \alpha(\delta) = -\frac{W(\delta) - W(v_{i^\dagger+1})}{\delta - v_{i^\dagger+1}} \\ i \in [i^\dagger + 1 : m-1],\ \delta \in [v_i, v_{i+1}]: \quad & \alpha(\delta) = -\frac{W(\delta) - W(v_i)}{\delta - v_i} \\ \delta \in [v_m, \infty): \quad & \alpha(\delta) = 0 \\ i \in [i^\dagger]: \quad & \eta(i) = \lambda_i \left(W(\delta^\dagger) + (\delta^\dagger - v_i)\alpha(\delta^\dagger)\right) \\ i \in [i^\dagger + 1 : m]: \quad & \eta(i) = \lambda_i W(v_i) \end{aligned}$$

Complementary slackness. We now argue the complementary slackness of the constructed assignment. Namely, for each state i and $\delta \in (-\infty, \infty)$ such that $\pi_i^*(\delta) > 0$, its corresponding dual constraint holds with equality, i.e.,

$$W(\delta) + (\delta - v_i)\alpha(\delta) = \frac{\eta(i)}{\lambda_i} \tag{3.1}$$

We verify this for each state $i \in [m]$ separately.

- Fix an arbitrary state $i \in [i^\dagger - 1]$, note that $\pi_i^*(\delta) > 0$ for $\delta = \delta^\dagger$ only. Here equality (3.1) holds by construction straightforwardly.
- Now consider threshold state $i^\dagger$, the same argument holds for equality (3.1) with $\pi_{i^\dagger}^*(\delta^\dagger) > 0$. It is remaining to verify equality (3.1) associated with $\pi_{i^\dagger}^*(v_{i^\dagger}) = 1 - p^\dagger > 0$. When $p^\dagger < 1$, constraint `complementary-slackness` in program $\mathcal{P}_{\texttt{SISU-OPT}}$ ensures that $\delta^\ddagger = v_{i^\dagger}$. Thus,

  $$W(v_{i^\dagger}) + (v_{i^\dagger} - v_{i^\dagger})\alpha(v_{i^\dagger}) \overset{(a)}{=} W(\delta^\ddagger) \overset{(b)}{=} W(\delta^\dagger) + (\delta^\dagger - \delta^\ddagger)\alpha(\delta^\dagger) \overset{(c)}{=} W(\delta^\dagger) + (\delta^\dagger - v_{i^\dagger})\alpha(\delta^\dagger) \overset{(d)}{=} \frac{\eta(i)}{\lambda_i}$$

  where equalities (a), (c) hold since $v_{i^\dagger} = \delta^\ddagger$; equality (b) holds due to constraint `dual-feasibility-1` in program $\mathcal{P}_{\texttt{SISU-OPT}}$ and the construction of $\alpha(\delta^\dagger)$; and equality (d) holds since equality (3.1) holds for $\pi_{i^\dagger}^*(\delta^\dagger)$ shown above.
- Fix an arbitrary state $i \in [i^\dagger + 1 : m]$, note that $\pi_i^*(\delta) > 0$ for $\delta = v_i$ only. Here equality (3.1) holds by construction straightforwardly.

Dual feasibility. To verify whether the dual constraints associated with $\pi_i^*(\delta)$ for state $i \in [i^\dagger]$ hold, note that

$$\frac{\eta(i)}{\lambda_i} \overset{(a)}{=} W(\delta^\dagger) + (\delta^\dagger - v_i)\alpha(\delta^\dagger) \overset{(b)}{=} W(\delta^\dagger) - (\delta^\dagger - v_i)W'(\delta^\dagger)$$

where equality (a) holds by the complementary slackness of $\pi_i^*(\delta^\dagger)$ verified above; and equality (b) holds by the construction of $\alpha(\delta^\dagger)$. Thus, we can rewrite those dual constraints associated with $\pi_i^*(\delta)$ for state $i \in [i^\dagger]$ as

$$W(\delta) + (\delta - v_i)\alpha(\delta) \le W(\delta^\dagger) - (\delta^\dagger - v_i)W'(\delta^\dagger) \tag{3.2}$$

To verify whether the dual constraints associated with $\pi_i^*(\delta)$ for state $i \in [i^\dagger + 1 : m]$ hold, by the complementary slackness of $\pi_i^*(\delta^\dagger)$ verified above, we can rewrite the dual constraints for state $i \in [i^\dagger + 1 : m]$ as

$$W(\delta) + (\delta - v_i)\alpha(\delta) \le W(v_i) \tag{3.3}$$

We verify both inequality (3.2) and inequality (3.3) for different values of δ in four cases separately: $\delta \in (-\infty, \delta^\ddagger]$; $\delta \in [\delta^\ddagger, v_{i^\dagger+1}]$; $\delta \in [v_j, v_{j+1}]$ for some $j \in [i^\dagger + 1 : m - 1]$; and $\delta \in [v_m, \infty)$. The argument mainly uses the curvature of function $W(\cdot)$ and the constraints in feasibility program $\mathcal{P}_{\texttt{SISU-OPT}}$, and the feasibility of the constructed dual assignment is summarized as follows:

Lemma 3.2. *The constructed dual assignment is a feasible solution to the dual program $\mathcal{P}_{\texttt{OPT-Dual}}$.*

In the main text below, we present the analysis for state $i \in [i^\dagger]$ for the first two cases, together with a graphical illustration of our argument (see Figure 2). The analysis of the third case is similar to the second case, and the fourth case is trivial. Therefore, we defer the later two cases and the analysis for the state $i \in [i^\dagger + 1, m]$ to Appendix D.4.

- Fix an arbitrary $\delta \in (-\infty, \delta^\ddagger]$. We illustrate this case in Figure 2a. Note that

$$\begin{aligned} W(\delta) + (\delta - v_i)\alpha(\delta) &\overset{(a)}{=} W(\delta) - (\delta - v_i)W'(\delta^\dagger) \\ &= W(\delta^\dagger) - (\delta^\dagger - v_i)W'(\delta^\dagger) + \big(W(\delta) - W(\delta^\dagger)\big) - (\delta - \delta^\dagger)W'(\delta^\dagger) \end{aligned}$$

where equality (a) holds due to the construction of $\alpha(\delta)$. Hence, to show inequality (3.2) in this case, it is sufficient to argue that

$$(\delta^\dagger - \delta)W'(\delta^\dagger) \le W(\delta^\dagger) - W(\delta) \tag{3.4}$$

which is true due to the curvature of function $W(\cdot)$. Specifically, if $\delta \in (-\infty, 0]$, inequality (3.4) holds since function $W(\cdot)$ is concave in $(-\infty, 0]$; if $\delta \in [0, \delta^\ddagger]$, inequality (3.4) holds since

$$W'(\delta^\dagger) \overset{(a)}{=} \frac{W(\delta^\ddagger) - W(\delta^\dagger)}{\delta^\ddagger - \delta^\dagger} \overset{(b)}{\ge} \frac{W(\delta) - W(\delta^\dagger)}{\delta - \delta^\dagger}$$

where equality (a) holds due to constraint `dual-feasibility-1` in program $\mathcal{P}_{\texttt{SISU-OPT}}$; and inequality (b) holds due to the convexity of function $W(\cdot)$ on $[0, \infty)$.

- Fix an arbitrary $\delta \in [\delta^\ddagger, v_{i^\dagger+1}]$. We illustrate this case in Figure 2b. By construction, $\alpha(\delta) = -(W(\delta) - W(v_{i^\dagger+1}))/(\delta - v_{i^\dagger+1})$. After rearranging the terms, inequality (3.2) becomes[16]

$$-\frac{W(\delta) - W(v_{i^\dagger+1})}{\delta - v_{i^\dagger+1}} \le -\frac{W(\delta) - \big(W(\delta^\dagger) - W'(\delta^\dagger)(\delta^\dagger - v_i)\big)}{\delta - v_i} \tag{3.5}$$

Here we argue that it is sufficient to show inequality (3.5) holds when we replace v_i with $\delta^\ddagger \ge v_i$. To see this, note that the right-hand side of inequality (3.5) is monotone decreasing as a function of v_i. In particular, consider function $f(x) \triangleq -(W(\delta) - (W(\delta^\dagger) - W'(\delta^\dagger)(\delta^\dagger - x)))/(\delta - x)$, and compute its derivative $f'(x) = -\frac{W(\delta) - W(\delta^\dagger) - W'(\delta^\dagger)(\delta - \delta^\dagger)}{(\delta - x)^2} \le 0$ where the last inequality holds since $W'(\delta^\dagger)(\delta - \delta^\dagger) \le W(\delta) - W(\delta^\dagger)$ if $\delta \ge \delta^\ddagger$, which is implied by constraint `dual-feasibility-1` and the convexity of function $W(\cdot)$ on $[0, \infty)$. Hence,

$$\begin{aligned} -\frac{W(\delta) - \big(W(\delta^\dagger) - W'(\delta^\dagger)(\delta^\dagger - v_i)\big)}{\delta - v_i} &\overset{(a)}{\ge} f(\delta^\ddagger) = -\frac{W(\delta) - \big(W(\delta^\dagger) - W'(\delta^\dagger)(\delta^\dagger - \delta^\ddagger)\big)}{\delta - \delta^\ddagger} \\ &\overset{(b)}{=} -\frac{W(\delta) - W(\delta^\ddagger)}{\delta - \delta^\ddagger} \overset{(c)}{\ge} -\frac{W(\delta) - W(v_{i^\dagger+1})}{\delta - v_{i^\dagger+1}} \end{aligned}$$

where inequality (a) holds due to the monotonicity of function $f(\cdot)$; equality (b) holds due to constraint `dual-feasibility-1` in program $\mathcal{P}_{\texttt{SISU-OPT}}$; and inequality (c) holds due to the convexity of function $W(\cdot)$ on $[0, \infty)$. ∎

[16] Here we use the fact that $\delta - v_i \ge 0$ for every state $i \in [i^\dagger]$, since $\delta \ge \delta^\ddagger \ge v_{i^\dagger} \ge v_i$ where the second inequality holds due to constraint `dual-feasibility-2` in program $\mathcal{P}_{\texttt{SISU-OPT}}$.

3.2 Approximation Lower Bounds of Direct Signaling Schemes When the receiver is fully rational, the optimality of direct signaling schemes follows from the standard revelation principle (Kamenica and Gentzkow, 2011). However, when the receiver is boundedly rational, this standard revelation principle fails. In this subsection, we provide an approximation lower bound for direct signaling schemes in SISU environment for a boundedly rational receiver. The proof of Theorem 3.2 is straightforward and thus we defer it to Appendix D.5.

THEOREM 3.2. *In SISU environments, there exists a problem instance (Example D.1) such that for any direct signaling scheme π, it is $\Omega(m)$-approximation to the optimal signaling scheme.*

In Theorem 4.2, we also give an $O(m)$-approximation upper bound for direct signaling schemes, which shows the tightness of our result.

4 State Dependent Sender Utility (SDSU) Environments

In this section, we consider the *state dependent sender utility (SDSU)* environments where the sender's utility $\{u_i\}_{i\in[m]}$ depends on both the realized state as well as the action of the receiver.[17] Recall that or a fully rational receiver, Lemma 2.1 shows the optimality of both censorship signaling schemes and direct signaling schemes. However, in SDSU environments, both censorship signaling schemes and direct signaling schemes are sub-optimal for a boundedly rational receiver. As the main result of this section, we first show that both censorship and direct signaling schemes are $\Omega(m)$-approximation (Proposition 4.1), and then we provide matching approximation upper bounds of censorship and direct signaling schemes (Theorem 4.2).

4.1 Approximation Lower Bounds of Censorship and Direct Signaling Schemes In this subsection, we provide approximation lower bounds for censorship and direct signaling schemes. In fact, we present a stronger result that quantifies the optimal payoff loss of a signaling scheme via its maximum number L of signals induced by each state.

THEOREM 4.1. *In SDSU environments, there exists a problem instance (Example 4.1) such that for any signaling scheme π with signal space Σ, it is $\Omega(m/L)$-approximation to the optimal signaling scheme, where $L \triangleq \max_{i\in[m]} |\{\delta \in \Sigma : \pi_i(\delta) > 0\}|$ denotes the maximum number of signals induced by a state in this signaling scheme π.*

The above result immediately implies approximation lower bounds for censorship and direct signaling schemes. [18]

PROPOSITION 4.1. *In SDSU environments, there exists a problem instance (Example 4.1) such that any censorship and any direct signaling scheme is $\Omega(m)$-approximation to the optimal signaling scheme.*

Proof. The above results follow from the definition of censorship/direct signaling schemes which have at most 2 signals induced from each state, namely, $L \le 2$, thus implying the results. □

Note that the $\Omega(m)$-approximation lower bound for censorship signaling schemes in SDSU environments (Proposition 4.1) stands in contrast to the optimality of censorship signaling schemes in SISU environments (Theorem 3.1).

Proof outline of Theorem 4.1. In the remainder of this subsection we outline the proof of Theorem 4.1 in three steps. All missing proofs from this subsection can be found in Appendix E.

Step 1- constructing problem instance and lower bounding the optimal payoff. We first construct a problem instance (Example 4.1) with m states and a carefully chosen bounded rationality level β that has the following properties: (i) the sender can only obtain utility from state m, i.e., $u_i > 0$ only when $i = m$; (ii) the prior probability for each state $i \in [m-1]$ is exponentially increasing with respect to the state. With the above

[17]Recall that the sender's utility is zero as long as the receiver takes action 0, and $u_i \ge 0$ denotes the sender's utility for realized state i and receiver taking action 1.

[18]Though it is not our focus, another broader class of signaling schemes that are studied in the literature is the monotone partitional signaling scheme (Kolotilin, 2018; Dworczak and Martini, 2019; Candogan, 2019). Both censorship signaling schemes and direct signaling schemes are also monotone partitional signaling schemes. A notably fact about monotone partitional signaling schemes is that each state can only induce at most 3 signals. Thus, Theorem 4.1 also implies that there exists a problem instance (Example 4.1) such that any monotone partitional signaling scheme is an $\Omega(m)$-approximate.

two properties, we are able to lower bound the optimal expected sender utility by $\Omega(K_1K_2m)$ where K_1, K_2 are problem-specific normalization terms (Lemma 4.1).

EXAMPLE 4.1. *Given an arbitrary $m \in \mathbb{N}_+$, consider a problem instance as follows: There are m states. The receiver has bounded rationality level β such that $^{\beta}/_{\log(\beta)} \geq 2m$. The sender utility $\{u_i\}$, the receiver utility difference $\{v_i\}$ are $u_i = \mathbb{1}[i = m], v_i = i, \forall i \in [m]$. Let $K_1 \triangleq {}^1/_{\sum_{i\in[m-1]}\exp(\beta i)}$. The prior $\{\lambda_i\}$ over state space $[m]$ is $\lambda_i = K_1K_2\left(m - i - \frac{1}{\beta}\right)\beta\exp(\beta i), \forall i \in [m-1]; \lambda_m = K_2$ where K_2 is the normalization term such that $\sum_{i\in[m]}\lambda_i = 1$.*

LEMMA 4.1. *In Example 4.1, the optimal expected sender utility $\mathbf{Payoff}[\pi^*] \geq \Omega(K_1K_2m)$.*

Step 2- upper bounding the payoff via censorship signaling schemes. In this step, we show that for any signaling scheme, we can upper bound expected sender utility in Example 4.1 via the utility from a set of censorship signaling schemes. In particular, for each state $i \in [m-1]$, given any possible pooling signal $\delta \in [v_i, v_m]$, we define following censorship signaling scheme where state i and state m are pooled on signal δ, and other states are fully revealed. Let $\delta^{\mathsf{avg}} \triangleq {}^{(\lambda_i i + \lambda_m m)}/_{(\lambda_i + \lambda_m)}$ be the pooling signal which state i and state m are fully pooled together. We consider following censorship signaling scheme $\pi^{(i,\delta)}$: if $\delta \leq \delta^{\mathsf{avg}}$, signaling scheme $\pi^{(i,\delta)}$ admits the form as follows

$$\pi_i^{(i,\delta)}(\delta) = 1; \quad \pi_m^{(i,\delta)}(\delta) = \frac{\lambda_i}{\lambda_m}\frac{\delta - i}{m - \delta}; \quad \pi_m^{(i,\delta)}(m) = 1 - \frac{\lambda_i}{\lambda_m}\frac{\delta - i}{m - \delta}; \quad \pi_j^{(i,\delta)}(j) = 1 \ \forall j \neq i, m$$

and if $\delta \geq \delta^{\mathsf{avg}}$, signaling scheme $\pi^{(i,\delta)}$ admits the form as follows

$$\pi_i^{(i,\delta)}(i) = 1 - \frac{\lambda_m}{\lambda_i}\frac{m - \delta}{\delta - i}; \quad \pi_i^{(i,\delta)}(\delta) = \frac{\lambda_m}{\lambda_i}\frac{m - \delta}{\delta - i}; \quad \pi_m^{(i,\delta)}(\delta) = 1; \quad \pi_j^{(i,\delta)}(j) = 1 \forall j \neq i, m$$

Fix any signaling scheme π where the signals induced by state m are $\{\delta_\ell\}_{\ell\in L}$. By definition,

$$\mathbf{Payoff}[\pi] \leq \sum_{\ell\in[L]}\sum_{i\in[m-1]}\mathbf{Payoff}\left[\pi^{(i,\delta_\ell)}\right] .$$

Now it remains to upper bound $\mathbf{Payoff}\left[\pi^{(i,\delta_\ell)}\right]$ for each state $i \in [m-1]$ and each $\ell \in [L]$.

Step 3- upper bounding $\mathbf{Payoff}\left[\pi^{(i,\delta)}\right]$. In this step, we upper bound the expected sender utility under the signaling scheme $\pi^{(i,\delta)}$. We below provide two characterizations on the upper bound of the expected sender utility $\mathbf{Payoff}\left[\pi^{(i,\delta)}\right]$ (Lemma 4.2), depending on the value of pooling signal δ. The proof of this lemma is deferred to Appendix E.2.

LEMMA 4.2. *In Example 4.1, for any state $i \in [m-1]$, the expected sender utility $\mathbf{Payoff}\left[\pi^{(i,\delta)}\right] = O(K_1K_2)$ for any $\delta \in [i, i + {}^{m\log(\beta)}/_{\beta}]$; and $\mathbf{Payoff}\left[\pi^{(i,\delta)}\right] = o({}^{K_1K_2}/_{m})$ for any $\delta \in [i + {}^{m\log(\beta)}/_{\beta}, m]$.*

With the above two characterizations on $\mathbf{Payoff}\left[\pi^{(i,\delta)}\right]$, we are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $L' \triangleq |\{\delta \in \Sigma : \pi_m(\delta) > 0\}|$ be the number of signals induced by state m, and denote these L' signals as $\{\delta_\ell\}_{\ell\in[L']}$. For each $\ell \in [L']$, since $^{\beta}/_{\log(\beta)} \geq 2m$, there exists an most one state $j \in [m-1]$ such that $\delta_\ell \in [j, j + {}^{m\log(\beta)}/_{\beta}]$. Invoking Lemma 4.2, we know that $\sum_{i\in[m-1]}\mathbf{Payoff}\left[\pi^{(i,\delta_\ell)}\right] = O(K_1K_2)$. Thus, invoking Lemma 4.1, we have

$$\frac{\mathbf{Payoff}[\pi^*]}{\mathbf{Payoff}[\pi]} = \frac{\Omega(K_1K_2m)}{L' \cdot O(K_1K_2)} = \Omega\left(\frac{m}{L'}\right),$$

which concludes the proof for Theorem 4.1. □

4.2 Approximation Upper Bounds of Censorship and Direct Signaling Schemes In this subsection, we discuss the approximation upper bounds of censorship and direct signaling schemes. The approximation upper bounds we provide here are indeed tight according to the lower bounds we established in Section 4.1.

THEOREM 4.2. *In SDSU environments, for a boundedly rational receiver, there exists a censorship/direct signaling scheme that is an $O(m)$-approximation to the optimal signaling scheme.*

We would like to highlight that designing censorship or direct signaling scheme with $O(m)$-approximation is not in-hindsight straightforward. For example, even for a fully rational receiver, the approximation of the full/no-information revealing or the better of the two could be unbounded. To establish Theorem 4.2, we start with characterizing a 4-approximation signaling scheme that has desired structure properties – the sender's signal either reveals the true state, or randomizes the receiver's uncertainty on only two states, then we utilize the structure of this 4-approximation signaling scheme to show the existence of $O(m)$-approximation censorship/direct signaling schemes.

LEMMA 4.3. *In SDSU environments, for a boundedly rational receiver, there exists a 4-approximation signaling scheme using at most $2m$ signals, and it has the following two properties:*

(i) each signal $\sigma \in \Sigma^*$ is induced by at most two states, i.e., $|\mathsf{supp}(\mu(\sigma))| \le 2$;

(ii) each pair of states (i, j) is pooled at most one signal, i.e., $|\{\sigma \in \Sigma^* : \mathsf{supp}(\mu(\sigma)) = \{i, j\}\}| \le 1$.

Furthermore, at most m signals are induced by two distinct states, i.e., $|\{\sigma : |\mathsf{supp}(\mu(\sigma))| = 2\}| \le m$.[19]

We now provide intuitions of the two properties of the signal scheme characterized in Lemma 4.3. Property (i) ensures that whenever the receiver sees a signal, she can infer that the realized state must be one of two particular states. From a practical perspective, this property is beneficial to a boundedly rational receiver as it makes the receiver's state inference easier. From the sender's perspective, property (ii) ensures that, for any pair of states, the sender only needs to design at most one pooling signal. We provide a proof overview of Lemma 4.3 in the end of this subsection and defer its formal proof to Appendix E.4.

With the results in Lemma 4.3, we are now ready to prove the Theorem 4.2.

Proof of Theorem 4.2. We first prove that there always exists a censorship signaling scheme that is $O(m)$-approximation. Let $\pi^\dagger$ with signal space $\Sigma^\dagger$ be the signaling scheme stated in Lemma 4.3. We denote U_{ij} as the expected sender utility induced by each pair of state (i, j), i.e., $U_{ij} \triangleq \sum_{\sigma:\pi_i^\dagger(\sigma)>0 \wedge \pi_j^\dagger(\sigma)>0} (\lambda_i u_i \pi_i^\dagger(\sigma) + \lambda_j u_j \pi_j^\dagger(\sigma)) W(\sigma)$. Let $(i^*, j^*) = \arg\max_{(i,j)} U_{ij}$. Note that by definition, and the property (i) of signaling scheme $\pi^\dagger$ we have $\mathbf{Payoff}[\pi^\dagger] \le m \cdot U_{i^* j^*}$.

Consider a binary-state instance $\mathcal{I} = (\hat{m}, \{\hat{\lambda}_k\}\{\hat{v}_k\}, \{\hat{u}_k\})$ induced by pair of states (i^*, j^*), i.e.,

$$\hat{m} \leftarrow 2, \qquad \hat{v}_1 \leftarrow v_{i^*}, \qquad \hat{v}_2 \leftarrow v_{j^*}, \qquad \hat{u}_1 \leftarrow u_{i^*}, \qquad \hat{u}_2 \leftarrow u_{j^*},$$
$$\hat{\lambda}_1 \leftarrow \frac{\lambda_{i^*}}{\lambda_{i^*} + \lambda_{j^*}}, \qquad \hat{\lambda}_2 \leftarrow \frac{\lambda_{j^*}}{\lambda_{i^*} + \lambda_{j^*}}$$

It can be shown that the optimal signaling scheme for this binary-state instance is a censorship signaling scheme (see Lemma E.5 and its proof in Appendix E.5). Let $\pi^\ddagger$ be the signaling scheme which coincides with the optimal signaling scheme for this binary-state instance, and reveals all other states. By construction, $\pi^\ddagger$ is again a censorship, and the expected sender utility

$$m \cdot \mathbf{Payoff}[\pi^\ddagger] \overset{(a)}{\ge} m \cdot U_{i^* j^*} \overset{(b)}{\ge} \mathbf{Payoff}[\pi^\dagger] \overset{(c)}{\ge} \frac{1}{4} \cdot \mathbf{Payoff}[\pi^*]$$

where π^* is the optimal signaling scheme, (a) holds due to the construction of $\pi^\ddagger$; (b) holds due to the definition of (i^*, j^*); and (c) holds since $\pi^\dagger$ is a 4-approximation to the signaling scheme π^*.

The proof of the $O(m)$-approximation for direct signaling scheme follows the similar argument which utilizes the structure of the signaling scheme $\pi^\dagger$, and thus is deferred to Appendix E.3. □

[19] Recall property (i) requires that for every σ, $|\mathsf{supp}(\mu(\sigma))| \le 2$.

Before finishing this subsection, we provide a proof overview for Lemma 4.3, and we defer the formal proof to Appendix E.4. At a high-level, our proof consists of two main steps. In the first step, we show that within the subclass of signaling schemes satisfying properties (i) (ii) in Lemma 4.3, there exists a signaling scheme π^* using at most $O(m^2)$ signals and achieving the optimality over all signaling schemes. In the second step, we discuss how to construct the signaling scheme stated in Lemma 4.3 based on the optimal signaling scheme π^* identified in the first step. Specifically, we establish a connection to the fractional generalized assignment problem (Shmoys and Tardos, 1993). In particular, by leveraging those two properties (i) (ii), we construct a linear program $\mathcal{P}_{\texttt{SDSU-OPT}}$ based on the optimal signaling scheme identified in the first step. This linear program upperbounds the optimal expected sender utility and has the same formulation as the fractional generalized assignment problem. Shmoys and Tardos (1993) show that the optimal integral solution of program $\mathcal{P}_{\texttt{SDSU-OPT}}$ (which has at most m non-zero entries) is a 2-approximation to the optimal fractional solution (which may have at most m^2 non-zero entries). With this result, we then convert this optimal integral solution to a signaling scheme stated in Lemma 4.3, which has at most $2m$ signals, and is a 2-approximation to the objective value of the optimal integral solution.

5 Rationality-Robust Information Design

In practice, the sender may not be able to have (or require significant cost to learn) the perfect knowledge of a receiver's bounded rationality level. Motivated by this concern, we introduce *rationality-robust information design*, in which a signaling scheme (a.k.a., information structure) is designed for a receiver whose bounded rationality level is unknown. The goal is to identify *robust* signaling schemes – ones with good (multiplicative) rationality-robust approximation to the optimal signaling scheme that is tailored to the receiver's bounded rationality level.

Definition 5.1. *Fixing any problem instance $\mathcal{I} = (m, \{\lambda_i\}, \{v_i\}, \{u_i\})$, the* rationality-robust approximation ratio $\Gamma(\pi, \mathcal{B})$ *of a given signaling scheme π and a set of possible bounded rationality levels $\mathcal{B} \subseteq [0, \infty)$ is*

$$\Gamma(\pi, \mathcal{B}) \triangleq \max_{\beta\in\mathcal{B}} \frac{\mathbf{Payoff}_\beta[\texttt{OPT}(\beta)]}{\mathbf{Payoff}_\beta[\pi]}$$

where $\texttt{OPT}(\beta)$ is the optimal signaling scheme[20] *for a receiver with bounded rationality level β (characterized in Lemma 2.1, Theorem 3.1, Lemma E.1); and $\mathbf{Payoff}_\beta[\texttt{OPT}(\beta)]$ (resp. $\mathbf{Payoff}_\beta[\pi]$) is the expected sender utility of signaling scheme $\texttt{OPT}(\beta)$ (resp. π) for bounded rationality level β.*

In the above definition, the rationality-robust approximation ratio is defined in worst-case over the set $\mathcal{B}$ of possible bounded rationality levels. Ideally, one would like to have a signaling scheme that is approximately optimal under any bounded rationality level, i.e., $\mathcal{B} = [0, \infty)$. This is the scenario illustrated in Section 5.1, in which we show that in SISU environments, the optimal censorship signaling scheme for a fully rational receiver can achieve 2 rationality-robust approximation for any receiver's bounded rationality level (Theorem 5.1). This suggests that, up to a two factor, the knowledge of the bounded rationality level are unimportant in SISU environments; and directly optimizing under fully rational receiver model is robust enough. In contrast, as we show in Section 5.2, there exists no signaling scheme with bounded rationality-robust approximation ratio in SDSU environments, when the sender has no knowledge of the receiver's bounded rationality level (Theorem 5.2). This impossibility result indicates that (a) there exists a tradeoff between the knowledge of the receiver's rationality level and the achievable rationality-robustness; and (b) even if the adversary is restricted to pick receiver's behavior in the quantal response model, designing robust signaling scheme still requires additional knowledge. Finally, we show a preliminary positive result in SDSU environments: for problem instances with binary state, when the actual rationality robust level is sufficiently large, learning the bounded rationality level up to a multiplicative error enables the sender to design signaling schemes with good rationality-robust approximation guarantee.

5.1 Rationality-Robust Signaling Schemes in SISU Environments In SISU environments, we show that for any problem instance, the optimal censorship signaling scheme (defined in Lemma 2.1) for a fully rational receiver achieves a 2 rationality-robust approximation when the sender has no knowledge of the receiver's bounded rationality level. We also provide an example to show the tightness of the result.

[20]Here we write the optimal signaling scheme with bounded rationality level β as $\texttt{OPT}(\beta)$, instead of π^* in previous sections, to emphasize its dependency on the rationality level β.

Theorem 5.1. *In SISU environments, for any problem instance, the optimal censorship signaling scheme $\hat{\pi}^*$ (defined in Lemma 2.1) for a fully rational receiver has rationality-robust approximation ratio $\Gamma(\hat{\pi}^*, [0, \infty)) \le 2$.*

To understand the intuition behind the above theorem, recall that the structural property of optimal censorship signaling scheme established in Proposition 3.1: The optimal censorship for a less rational receiver requires the sender to reveal more information. As a result, the optimal censorship for receiver with $\beta = \infty$ reveals least information and pools most states compared to other optimal censorship for receiver with $\beta < \infty$. Meanwhile, the pooling signal $\hat{\delta}^\dagger \equiv 0$ in $\hat{\pi}^*$ ensures that the utility contributed from those pooled states is at least half of the utility contributed from those states in the optimal censorship with less rational receiver.

Proof of Theorem 5.1. Fix any bounded rationality level $\beta \in \mathcal{B}$. For signaling scheme $\hat{\pi}^*$,

$$\begin{aligned}\textbf{Payoff}_\beta[\hat{\pi}^*] &= \sum_{i \in [\hat{i}^\dagger - 1]} \lambda_i W(0) + \lambda_{\hat{i}^\dagger} \left(\hat{p}^\dagger W(0) + \left(1 - \hat{p}^\dagger\right) W\left(v_{\hat{i}^\dagger}\right)\right) + \sum_{i \in [\hat{i}^\dagger + 1 : m]} \lambda_i W(v_i) \\ &\ge \sum_{i \in [\hat{i}^\dagger - 1]} \lambda_i W(0) + \sum_{i \in [\hat{i}^\dagger : m]} \lambda_i W(v_i)\end{aligned}$$

where $\hat{i}^\dagger, \hat{p}^\dagger$ is the threshold state, the threshold state probability of $\hat{\pi}^*$.[21] Moreover, by Theorem 3.1, the optimal expected sender utility under the bounded rationality level β is

$$\begin{aligned}\textbf{Payoff}_\beta[\texttt{OPT}(\beta)] &= \sum_{i \in [i^\dagger - 1]} \lambda_i W(\delta^\dagger) + \lambda_{i^\dagger} \left(p^\dagger W(\delta^\dagger) + \left(1 - p^\dagger\right) W\left(v_{i^\dagger}\right)\right) + \sum_{i \in [i^\dagger + 1 : m]} \lambda_i W(v_i) \\ &\le \sum_{i \in [i^\dagger]} \lambda_i W(\delta^\dagger) + \sum_{i \in [i^\dagger + 1 : m]} \lambda_i W(v_i)\end{aligned}$$

Recall Proposition 3.1 implies that $\hat{i}^\dagger \ge i^\dagger$. Hence,

$$\frac{\textbf{Payoff}_\beta[\texttt{OPT}(\beta)]}{\textbf{Payoff}_\beta[\hat{\pi}^*]} \le \max\left\{\max_{i \in [i^\dagger]} \frac{\lambda_i W(\delta^\dagger)}{\lambda_i W(0)}, \max_{i \in [i^\dagger + 1 : \hat{i}^\dagger - 1]} \frac{\lambda_i W(v_i)}{\lambda_i W(0)}, \max_{i \in [\hat{i}^\dagger : m]} \frac{\lambda_i W(v_i)}{\lambda_i W(v_i)}\right\} = \frac{W(\delta^\dagger)}{W(0)} \overset{(a)}{\le} 2$$

where inequality (a) holds since $W(0) = 1/2 \ge W(\delta)/2$ for all $\delta \in (-\infty, \infty)$. □

The below result (its proof is deferred to Appendix F.1) shows the tightness of the robust-rationality approximation ratio established in Theorem 5.1.

Proposition 5.1. *In SISU environments, for any $\varepsilon > 0$, there exists a problem instance such that the optimal censorship $\hat{\pi}^*$ (defined in Lemma 2.1) for a fully rational receiver has rationality-robust approximation ratio $\Gamma(\hat{\pi}^*, [0, \infty)) \ge 2 - \varepsilon$.*

We conclude this subsection by noting that the robust signaling scheme $\hat{\pi}^*$ used in Theorem 5.1 is the optimal censorship for a fully rational receiver. However, as we show in Proposition 5.2 below (its proof is straightforward and is deferred to Appendix F.2), the optimal direct signaling scheme $\widetilde{\pi}^*$ (defined in Lemma 2.1) for a fully rational receiver cannot achieve any meaningful rationality-robust approximation guarantee. This again mirrors the analogous separation results on the censorship and direct signaling schemes we show in previous sections.

Proposition 5.2. *In SISU environments, there exists a problem instance such that the optimal direct signaling scheme $\widetilde{\pi}^*$ for a fully rational receiver has rationality-robust approximation ratio $\Gamma(\widetilde{\pi}^*, [0, \infty)) = \infty$.*

[21] Here we use the superscript ^ to denote the concepts in signaling scheme $\hat{\pi}^*$.

5.2 Rationality-Robust Signaling Schemes in SDSU Environments Unlike SISU environments where the knowledge of the rationality level is unimportant up to a two factor (Theorem 5.1), in this subsection, we first present the following negative result that without the knowledge of the rationality level, there exists no signaling scheme with bounded rationality-robust approximation ratio, even if the state space is binary (Theorem 5.2). Nonetheless, we also provide a positive result for binary-state problem instances under a reasonable condition of receiver's bounded rationality levels (Proposition F.1).

THEOREM 5.2. *In SDSU environments, there exists a problem instance (Example 5.2) with binary state such that for any signaling scheme π and any $\beta_0 \geq 0$, the rationality-robust approximation ratio with respect to $\mathcal{B} = [\beta_0, \infty)$ is unbounded, i.e., $\Gamma(\pi, [\beta_0, \infty)) = \infty$.*

Proof overview of Theorem 5.2. The formal proof of Theorem 5.2 is deferred to Appendix F.3. Here we sketch the high-level idea behind the proof. Our proof proceeds with two steps as follows. In the first step, we construct a binary-state problem instance in Example 5.2. We further provide a finite set $\mathcal{B} \triangleq \{\beta_\ell\}_{\ell \in [L]}$ where[22] $\beta_\ell \triangleq L^\ell$. Recall that when the state space is binary, the optimal signaling scheme is a censorship signaling scheme (Lemma E.5). The construction in Example 5.2 ensures that the contribution in the optimal (censorship) signaling scheme mainly comes from the pooling signal $\delta^\dagger$. Similar to the analysis in Theorem 4.1, the value of $\delta^\dagger$ is quite sensitive to the rationality level β_ℓ. As a consequence, for any $\beta_1, \beta_2 \in \mathcal{B}$ such that $\beta_1 \neq \beta_2$, it satisfies that $\mathbf{Payoff}_{\beta_2}[\texttt{OPT}(\beta_1)] \ll \mathbf{Payoff}_{\beta_2}[\texttt{OPT}(\beta_2)]$, which says that the optimal signaling scheme under a specific rationality level must have a very bad performance if sender implements such optimal signaling scheme with a receiver who has a different bounded rationality level.

EXAMPLE 5.2. *Consider the following problem instance with binary state (i.e., $m = 2$),*

$$\lambda_1 = \frac{1}{2}, \qquad \lambda_2 = \frac{1}{2}, \qquad v_1 = 1, \qquad v_2 = 2, \qquad u_1 = 0, \qquad u_2 = 1.$$

In the second step, given the above constructed binary-state problem instance and the set $\mathcal{B}$ of rationality levels, we introduce the following *factor-revealing program* to lower bound the optimal rationality-robust approximation ratio, i.e., $\min_\pi \Gamma(\pi, \mathcal{B})$.

$$\begin{array}{lll}
\min\limits_{\boldsymbol{\pi} \geq \mathbf{0}, \Gamma \geq 0} & \Gamma & \text{s.t.} \\
& \lambda_1 \pi_1(\delta) \cdot (\delta - v_1) + \lambda_2 \pi_2(\delta) \cdot (\delta - v_2) \geq 0 & \delta \in (-\infty, \infty) \\
& \displaystyle\int_{-\infty}^{\infty} \pi_i(\delta) d\delta = 1 & i \in [2] \\
& \pi_i(\delta) \geq 0 & \delta \in (-\infty, \infty), \ i \in [2] \\
& \mathbf{Payoff}_{\beta_\ell}[\pi] \geq \frac{1}{\Gamma} \frac{1}{\beta_\ell \exp(\beta_\ell)}, & \ell \in [L]
\end{array}$$

In this program, the variables $\boldsymbol{\pi}$ can be interpreted as a signaling scheme, and Γ can be interpreted as its rationality-robust approximation ratio. In particular, the last constraint requires the expected sender utility of signaling scheme π for a receiver with bounded rationality level β_ℓ is at least a Γ-approximation to ${}^{1}/_{\beta_\ell \exp(\beta_\ell)}$, which, as we show in the proof, is a lower bound of the optimal expected sender utility $\mathbf{Payoff}_{\beta_\ell}[\texttt{OPT}(\beta_\ell)]$. Notably, this program is essentially a linear program. Hence, by explicitly constructing a dual assignment in its dual program and then invoking the weak duality, we can lower bound its optimal objective value by $\Omega(L)$. Finally, setting L to be infinite finishes the proof of Theorem 5.2.

Positive result for binary-state instances in SDSU environments. Theorem 5.2 highlights the importance of the knowledge of the receiver's bounded rationality level in SDSU environments. Namely, even there are *only* two states, if the sender does not have any knowledge about receiver's bounded rationality level, then it is impossible to hope for a robust signaling scheme that would have bounded rationality-robust approximation ratio.

In Appendix F.4, we present a positive result for problem instances with binary states (see Proposition F.1 and its the proof in Appendix F.4), which shows that if the sender learns the receiver's bounded rationality level

[22]Here L is a sufficiently large constant, which goes to infinite in the end of the analysis.

up to a multiplicative error $K \geq 1$, i.e., $\mathcal{B} = [\beta_0, K\beta_0]$, and β_0 is larger than an instance-dependent bound,[23] then there exists (censorship) signaling schemes whose rationality-robust approximation ratio depends linearly on multiplicative error K.

6 Conclusions and Future Important Directions

In this work, we develop a theory of rationality-robust information design in the canonical setting of Bayesian persuasion with binary receiver action. We first identify conditions under which the optimal signaling scheme structure for a fully rational receiver remains optimal or approximately optimal for a boundedly rational receiver. We then study the existence and construction of robust signaling schemes when there is uncertainty about the receiver's bounded rationality level. Below we highlight the following natural and important directions of future research.

The most general direction from this paper is to develop a theory of information design or mechanism design for agents with bounded rationality. Most existing results on this direction restrict attention to specific problems (see Appendix A for more details). An interesting question is whether there exist conditions under which the optimal/approximately optimal results for fully rational agents extend to boundedly rational agents under a broad class of information/mechanism design problems. For agents with bounded rationality, the standard revelation principle fails, and it is no longer without loss of generality to impose incentive compatibility. In this sense, the bounded rationality also provides a motivation and new perspective on the recent literature on non-truthful mechanism design (e.g., Feng and Hartline, 2018; Cai et al., 2019; Daskalakis et al., 2020; Assadi et al., 2022).

The bounded rationality specifies how agents select their actions. Therefore, similar to our findings, mechanisms that are equivalent under fully rationality (e.g., second-price auction and English auction) may lead to different outcomes under bounded rationality. Exploring our first question in mechanism design context may provide an alternative justification on practical preference of certain mechanisms format (cf. Akbarpour and Li, 2020). For information design problems, action sets of agents are given exogenously. In contrast, for mechanism design problems, action sets for agents are usually designed endogenously. Thus, it is also interesting to systematically develop theory to understand how to design action sets (a.k.a., mechanism formats) and preference over classic format.

For our Bayesian persuasion problem, there are also several interesting open questions. In SISU environments, a natural question is whether there exists a signaling scheme that can beat the 2 rationality-robust approximation ratio achieved by the optimal censorship for the fully rational receiver. In SDSU environments, one immediate question is whether there exists a robust signaling scheme for problem instances with multiple states under a reasonable boundedness condition on the receiver's bounded rationality levels. More importantly, what is the fine-grained tradeoff between the knowledge on receiver's behavior and the achievable rationality-robustness of signaling schemes? Conceptually, these questions share similar flavor with the prior-independent mechanism design framework (e.g., Dhangwatnotai et al., 2015; Fu et al., 2015; Allouah and Besbes, 2020; Hartline et al., 2020).

Finally, another direction of interest is to characterize the computational complexity of computing an optimal (or approximately optimal) signaling scheme in different environments. Note that when the receiver is fully rational, the optimal signaling scheme can be computed in polynomial time. When the receiver is boundedly rational, in Appendix G, we present some preliminary results on characterizing the complexity of computing the (approximately) optimal signaling scheme in both SISU/SDSU environments. Whether our results can be strengthened is an interesting and important future direction.

Acknowledgement

We thank the anonymous reviewers for the helpful comments. This work is supported in part by the Office of Naval Research under Grant N00014-20-1-2240 and a J.P. Morgan Faculty Research Award.

References

Mohammad Akbarpour and Shengwu Li. Credible auctions: A trilemma. *Econometrica*, 88(2):425–467, 2020.

[23]Recall our negative result (Theorem 5.2) shows that there exists no signaling scheme with finite rationality-robust approximation ratio with respect to $\mathcal{B} = [\beta_0, \infty)$ for any $\beta_0 \geq 0$. It remains as an open question whether similar rationality-robust signaling schemes exists for small β_0.

Amine Allouah and Omar Besbes. Prior-independent optimal auctions. *Management Science*, 66(10):4417–4432, 2020.

Ricardo Alonso and Odilon Câmara. Political disagreement and information in elections. *Games and Economic Behavior*, 100:390–412, 2016a.

Ricardo Alonso and Odilon Câmara. Persuading voters. *American Economic Review*, 106(11):3590–3605, 2016b.

Edward J Anderson and Peter Nash. *Linear programming in infinite-dimensional spaces: theory and applications*. John Wiley & Sons, 1987.

Jerry Anunrojwong, Krishnamurthy Iyer, and David Lingenbrink. Persuading risk-conscious agents: A geometric approach. *Available at SSRN 3386273*, 2020.

Itai Arieli and Yakov Babichenko. Private bayesian persuasion. *Journal of Economic Theory*, 182:185–217, 2019.

Sepehr Assadi, Hrishikesh Khandeparkar, Raghuvansh R Saxena, and S Matthew Weinberg. Separating the communication complexity of truthful and nontruthful algorithms for combinatorial auctions. *SIAM Journal on Computing*, (0):STOC20–75, 2022.

Yunus C Aybas and Eray Turkel. Persuasion with coarse communication. *arXiv preprint arXiv:1910.13547*, 2019.

Yakov Babichenko and Siddharth Barman. Algorithmic aspects of private bayesian persuasion. In *8th Innovations in Theoretical Computer Science Conference*, 2017.

Yakov Babichenko, Inbal Talgam-Cohen, Haifeng Xu, and Konstantin Zabarnyi. Regret-minimizing bayesian persuasion. In *Proceedings of the 22nd ACM Conference on Economics and Computation*, pages 128–128, 2021.

Ashwinkumar Badanidiyuru, Kshipra Bhawalkar, and Haifeng Xu. Targeting and signaling in ad auctions. In *Proceedings of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2545–2563, 2018.

Dirk Bergemann and Stephen Morris. Information design: A unified perspective. *Journal of Economic Literature*, 57(1):44–95, 2019.

Dirk Bergemann, Yang Cai, Grigoris Velegkas, and Mingfei Zhao. Is selling complete information (approximately) optimal? In *Proceedings of the 23rd ACM Conference on Economics and Computation*, pages 608–663, 2022a.

Dirk Bergemann, Tibor Heumann, Stephen Morris, Constantine Sorokin, and Eyal Winter. Optimal information disclosure in auctions. *American Economic Review: Insights*, 2022b.

David Blackwell. Equivalent comparisons of experiments. *The annals of mathematical statistics*, pages 265–272, 1953.

Alexander W Bloedel and Ilya Segal. Persuading a rationally inattentive agent. Technical report, Working paper, 2020.

Mark Braverman, Jieming Mao, Jon Schneider, and Matt Weinberg. Selling to a no-regret buyer. In *Proceedings of the 19th ACM Conference on Economics and Computation*, pages 523–538, 2018.

Linda Cai, Clayton Thomas, and S Matthew Weinberg. Implementation in advised strategies: Welfare guarantees from posted-price mechanisms when demand queries are np-hard. *arXiv preprint arXiv:1910.04342*, 2019.

Modibo K Camara, Jason D Hartline, and Aleck Johnsen. Mechanisms for a no-regret agent: Beyond the common prior. In *2020 IEEE 61st annual symposium on foundations of computer science*, pages 259–270, 2020.

Ozan Candogan. Optimality of double intervals in persuasion: A convex programming framework. *Available at SSRN 3452145*, 2019.

Matteo Castiglioni, Andrea Celli, Alberto Marchesi, and Nicola Gatti. Online bayesian persuasion. *Advances in Neural Information Processing Systems*, 33:16188–16198, 2020.

Matteo Castiglioni, Alberto Marchesi, Andrea Celli, and Nicola Gatti. Multi-receiver online bayesian persuasion. In *International Conference on Machine Learning*, pages 1314–1323. PMLR, 2021.

Shuchi Chawla, Kira Goldner, J Benjamin Miller, and Emmanouil Pountourakis. Revenue maximization with an uncertainty-averse buyer. In *Proceedings of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2050–2068, 2018.

Shuchi Chawla, Nikhil R Devanur, Anna R Karlin, and Balasubramanian Sivan. Simple pricing schemes for consumers with evolving values. *Games and Economic Behavior*, 2022.

Yiling Chen and Tao Lin. Persuading a behavioral agent: Approximately best responding and learning. *arXiv preprint arXiv:2302.03719*, 2023.

Geoffroy de Clippel and Xu Zhang. Non-bayesian persuasion. 2022.

Flavin Cristian. Understanding fault-tolerant distributed systems. *Communications of the ACM*, 34(2):56–78, 1991.

Rachel Cummings, Nikhil R Devanur, Zhiyi Huang, and Xiangning Wang. Algorithmic price discrimination. In *Proceedings of the 14th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2432–2451, 2020.

Constantinos Daskalakis, Maxwell Fishelson, Brendan Lucier, Vasilis Syrgkanis, and Santhoshini Velusamy. Simple, credible, and approximately-optimal auctions. In *Proceedings of the 21st ACM Conference on Economics and Computation*, pages 713–713, 2020.

Yuan Deng, Jon Schneider, and Balasubramanian Sivan. Strategizing against no-regret learners. *Advances in neural information processing systems*, 32, 2019a.

Yuan Deng, Jon Schneider, and Balasubramanian Sivan. Prior-free dynamic auctions with low regret buyers. *Advances in Neural Information Processing Systems*, 32, 2019b.

Peerapong Dhangwatnotai, Tim Roughgarden, and Qiqi Yan. Revenue maximization with a single sample. *Games and Economic Behavior*, 91:318–333, 2015.

Shaddin Dughmi. Algorithmic information structure design: a survey. *ACM SIGecom Exchanges*, 15(2):2–24, 2017.

Shaddin Dughmi and Yuval Peres. Mechanisms for risk averse agents, without loss. *arXiv preprint arXiv:1206.2957*, 2012.

Shaddin Dughmi and Haifeng Xu. Algorithmic bayesian persuasion. In *Proceedings of the forty-eighth annual ACM symposium on Theory of Computing*, pages 412–425, 2016.

Shaddin Dughmi and Haifeng Xu. Algorithmic persuasion with no externalities. In *Proceedings of the 18th ACM Conference on Economics and Computation*, pages 351–368, 2017.

Shaddin Dughmi, David Kempe, and Ruixin Qiang. Persuasion with limited communication. In *Proceedings of the 17th ACM Conference on Economics and Computation*, pages 663–680, 2016.

Piotr Dworczak and Giorgio Martini. The simple economics of optimal persuasion. *Journal of Political Economy*, 127(5):1993–2048, 2019.

Piotr Dworczak and Alessandro Pavan. Preparing for the worst but hoping for the best: Robust (bayesian) persuasion. *Econometrica*, 90(5):2017–2051, 2022.

David Easley and Arpita Ghosh. Behavioral mechanism design: Optimal crowdsourcing contracts and prospect theory. In *Proceedings of the 16th ACM Conference on Economics and Computation*, pages 679–696, 2015.

Yuval Emek, Michal Feldman, Iftah Gamzu, Renato PaesLeme, and Moshe Tennenholtz. Signaling schemes for revenue maximization. *ACM Transactions on Economics and Computation*, 2(2):1–19, 2014.

Yiding Feng and Jason D Hartline. An end-to-end argument in mechanism design (prior-independent auctions for budgeted agents). In *2018 IEEE 59th Annual Symposium on Foundations of Computer Science*, pages 404–415, 2018.

Yiding Feng, Wei Tang, and Haifeng Xu. Online bayesian recommendation with no regret. In *Proceedings of the 23rd ACM Conference on Economics and Computation*, page 818–819, 2022.

Hu Fu, Jason Hartline, and Darrell Hoy. Prior-independent auctions for risk-averse agents. In *Proceedings of the 14th ACM conference on Electronic commerce*, pages 471–488, 2013.

Hu Fu, Nicole Immorlica, Brendan Lucier, and Philipp Strack. Randomization beats second price as a prior-independent auction. In *Proceedings of the 16th ACM Conference on Economics and Computation*, pages 323–323, 2015.

Jiarui Gan, Minbiao Han, Jibang Wu, and Haifeng Xu. Robust stackelberg equilibria. *arXiv preprint arXiv:2304.14990*, 2023.

Ronen Gradwohl, Niklas Hahn, Martin Hoefer, and Rann Smorodinsky. Algorithms for persuasion with limited communication. *Mathematics of Operations Research*, 2022.

Yingni Guo and Eran Shmaya. The interval structure of optimal disclosure. *Econometrica*, 87(2):653–675, 2019.

Jason Hartline, Aleck Johnsen, and Yingkai Li. Benchmark design and prior-independent optimization. In *2020 IEEE 61st Annual Symposium on Foundations of Computer Science*, pages 294–305, 2020.

Ju Hu and Xi Weng. Robust persuasion of a privately informed receiver. *Economic Theory*, 72:909–953, 2021.

Emir Kamenica. Bayesian persuasion and information design. *Annual Review of Economics*, 11:249–272, 2019.

Emir Kamenica and Matthew Gentzkow. Bayesian persuasion. *American Economic Review*, 101(6):2590–2615, 2011.

Anton Kolotilin. Optimal information disclosure: A linear programming approach. *Theoretical Economics*, 13(2): 607–635, 2018.

Anton Kolotilin, Tymofiy Mylovanov, Andriy Zapechelnyuk, and Ming Li. Persuasion of a privately informed receiver. *Econometrica*, 85(6):1949–1964, 2017.

Anton Kolotilin, Timofiy Mylovanov, and Andriy Zapechelnyuk. Censorship as optimal persuasion. *Theoretical Economics*, 17(2):561–585, 2022.

Israel Koren and C Mani Krishna. *Fault-tolerant systems*. Morgan Kaufmann, 2020.

Svetlana Kosterina. Persuasion with unknown beliefs. *Theoretical Economics*, 17(3):1075–1107, 2022.

Jean-Claude Laprie. Dependability: Basic concepts and terminology. In *Dependability: Basic Concepts and Terminology*, pages 3–245. Springer, 1992.

Maël Le Treust and Tristan Tomala. Persuasion with limited communication capacity. *Journal of Economic Theory*, 184:104940, 2019.

David Lingenbrink and Krishnamurthy Iyer. Optimal signaling mechanisms in unobservable queues. *Operations research*, 67(5):1397–1416, 2019.

Elliot Lipnowski, Laurent Mathevet, and Dong Wei. Attention management. *American Economic Review: Insights*, 2(1):17–32, 2020.

Yishay Mansour, Alex Slivkins, Vasilis Syrgkanis, and Zhiwei Steven Wu. Bayesian exploration: Incentivizing exploration in bayesian games. *Operations Research*, 70(2):1105–1127, 2022.

Filip Matějka and Alisdair McKay. Rational inattention to discrete choices: A new foundation for the multinomial logit model. *American Economic Review*, 105(1):272–98, 2015.

Richard D McKelvey and Thomas R Palfrey. Quantal response equilibria for normal form games. *Games and economic behavior*, 10(1):6–38, 1995.

Jérôme Renault, Eilon Solan, and Nicolas Vieille. Optimal dynamic information provision. *Games and Economic Behavior*, 104:329–349, 2017.

John Rust. Optimal replacement of gmc bus engines: An empirical model of harold zurcher. *Econometrica: Journal of the Econometric Society*, pages 999–1033, 1987.

David B Shmoys and Éva Tardos. An approximation algorithm for the generalized assignment problem. *Mathematical programming*, 62(1):461–474, 1993.

Kalyan Talluri and Garrett Van Ryzin. Revenue management under a general discrete choice model of consumer behavior. *Management Science*, 50(1):15–33, 2004.

Wei Tang and Chien-Ju Ho. On the bayesian rational assumption in information design. In *Proceedings of the AAAI Conference on Human Computation and Crowdsourcing*, volume 9, pages 120–130, 2021.

Haifeng Xu. On the tractability of public persuasion with no externalities. In *Proceedings of the 14th Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2708–2727. SIAM, 2020.

Guanghui Yu, Wei Tang, Saumik Narayanan, and Chien-Ju Ho. Encoding human behavior in information design through deep learning. *Advances in neural information processing systems*, 2023.

A Further Related Work

Our work on relaxing rationality assumption in information design is conceptually similar to a large literature in mechanism design without/relaxing rationality assumption. For example, Braverman et al. (2018) and follow-up works Deng et al. (2019a,b) study revenue-maximization for a single buyer who uses no-regret algorithm in a repeated game with a seller. Camara et al. (2020) study a repeated Stackelberg game where both players use no-regret learning algorithm. Behavioral mechanism design (Easley and Ghosh, 2015) study how departures from standard economic models of agent behavior affect mechanism design. Chawla et al. (2018) study the revenue-maximization when the buyer's behavioral model is beyond expected utility theory and characterize mechanism that is robust to the buyer's risk attitude. Other related works in mechanism design include Fu et al. (2013); Chawla et al. (2022); Dughmi and Peres (2012).

Our work relates to a rich literature on information design. Since the seminal work (Kamenica and Gentzkow, 2011) that setup the Bayesian persuasion problem that studies the game on strategic communication between a sender and a receiver, the framework has inspired an active line of research in information design games (e.g., see the surveys by Dughmi, 2017; Kamenica, 2019; Bergemann and Morris, 2019). In addition to applications mentioned in introduction, Bayesian persuasion has also been studied in other different applications like online ad auction (Emek et al., 2014; Cummings et al., 2020; Arieli and Babichenko, 2019; Bergemann et al., 2022b), recommendation Mansour et al. (2022); Feng et al. (2022), and voting (Alonso and Câmara, 2016a,b). Our work extends this line of research by relaxing the standard rationality assumption. In particular, we consider a boundedly rational receiver by modeling her as a (logit) quantal response player, while standard framework usually assumes that the receiver is fully rational, i.e., an expected utility maximizer. Relaxing rationality assumptions has been studied in other information design literature. For example, Clippel and Zhang (2022) study how receiver's mistakes in probabilistic inference impact optimal persuasion, Anunrojwong et al. (2020) study a persuasion problem where the receiver's utility may be nonlinear in her belief, and Tang and Ho (2021); Yu et al. (2023) run behavioral experiments and relax the Bayesian rational assumption in a simple persuasion setting. Castiglioni et al. (2020, 2021) also relax traditional assumptions in an online setting. Our work also conceptually relates to recent papers that focus on settings where the receiver has limited attention to process and utilize the information (Lipnowski et al., 2020; Bloedel and Segal, 2020). Since it has been shown that the optimal stochastic response of a rationally inattentive receiver takes a "logit" form (Matějka and McKay, 2015), similar to our results in Section 3, Bloedel and Segal

(2020) show that the optimal information policy in SISU environments for inattentive receivers has a censorship structure. Our work differs from their work as we consider a more general sender payoff structure while the payoff in Bloedel and Segal (2020) depends linearly on the state. Moreover, in addition to characterizing the optimal information policy, we also study the design of approximately optimal and rationality-robust information polices for boundedly rational receivers in both SISU and SDSU environments. We also mention that our persuasion problem for the boundedly rational receiver is equivalent to a public persuasion problem (Dughmi and Xu, 2017; Xu, 2020) for a continuum population of *rational* receivers with a specific utility structure (see Appendix B.4 for more detailed discussions).

Our work has utilized and compared with censorship and direct signaling schemes. As a general class of signaling schemes, censorship has been studied in the recent literature. Kolotilin et al. (2022) consider the setting where the sender's utility depends only on the expected state. They show that a censorship is optimal if and only if the sender's marginal utility is quasi-concave. Kolotilin (2018) and Alonso and Câmara (2016a) provide sufficient conditions for the optimality of censorship in different contexts. Our paper departs from these works by not only considering the optimality of censorship signaling schemes under a different context (i.e., with boundedly rational receiver), but also studying its approximation guarantees when it is not optimal. Direct signaling scheme has also been studied in persuasion setting with binary action (Dughmi and Xu, 2017; Babichenko and Barman, 2017; Xu, 2020; Feng et al., 2022). On the other hand, signaling schemes like censorship in finite state space use at most m signals, and direct signaling schemes use at most 2 signals, Gradwohl et al. (2022) analyze optimal persuasion subject to limited signals constraint. However, neither of the two specific classes of problems they consider – symmetric instances and independent instances – is applicable to our problem, and thus cannot inform any approximation guarantees in our setting. Other related works on persuasion with limited communication constraint include Dughmi et al. (2016); Le Treust and Tomala (2019); Aybas and Turkel (2019).

B Motivating Example and Extensions

B.1 Motivating Examples in Section 1 In the example of product advertising (Emek et al., 2014; Arieli and Babichenko, 2019), a grocery store (i.e., sender), who observes the true product quality (i.e., state) with exogenous prices, performs advertising (i.e., signaling scheme) to a consumer (i.e., receiver) who makes binary purchasing decisions. In recommendation letter (Dughmi, 2017), an advisor, who observes the true ability of students, writes recommendation letter to a recruiter who makes binary hiring decision. In short video recommendation (Mansour et al., 2022; Feng et al., 2022), a short video platform (e.g., TikTok, Reels) who observes the content of short videos, makes recommendation to a user who decide to either watch or skip the video. In targeting in sponsored search (Badanidiyuru et al., 2018; Bergemann et al., 2022b), a search engine (e.g., Bing, Google) who observes the attribute of an impression (i.e., a match between advertiser and user), does targeting to an advertiser who decides bid or not bid for this impression.

In the examples of product advertising and recommendation letter, when the grocery store only cares about whether the buyer buys the product, and the advisor only cares whether the student is hired, the sender's utility is independent of the realized state. Under these scenarios, both examples can be formulated as SISU environments in our work. In the examples of short video recommendation and targeting in sponsored search, the sender's utility could depend on the realized state. For example, the short videos could be sponsored by some companies, and these sponsored videos might bring different revenue to the platform if the user chooses to watch the videos. Similarly, in targeting of sponsored search, different impressions could lead to different click-through rates, the revenue will be generated to the the search engine if the displayed advertising is clicked. Under these scenarios, both examples can be formulated as SDSU environments in our work.

B.2 Extensions on without Assuming $u_i(1) \ge u_i(0)$ By the definition of SISU environments, we would like to first note that this assumption (i.e., $u_i(1) \ge u_i(0)$ for all states $i \in [m]$) trivially holds in SISU environments. In SDSU environments, all our lower bound results (including the impossibility result in rationality-robust information design) also hold without this assumption. The preliminary positive result Proposition F.1 also holds via a similar duality argument. It would be an interesting future direction to explore whether Theorem 4.2 still holds without this assumption.

B.3 Extensions to General Quantal Response Curve W Our results in Section 3 and Section 4 on characterizing the optimal signaling schemes can be readily extended to a more general quantal response behavior

W. For example, the characterization on the optimality of censorship signaling scheme for SISU environments (i.e., Theorem 3.1), the structure characterization of the optimal signaling scheme for SDSU environments (i.e., Lemma E.1) hold as long as the function W is S-shaped.

B.4 Reinterpretation via Public Persuasion One explanation of a quantal response receiver is that she faces a action-specific random shock $\{\varepsilon(a)\}_{a\in\mathcal{A}}$ when she is making the decision (see Rust, 1987; McKelvey and Palfrey, 1995). In particular, given posterior belief $\mu \in \Delta([m])$, the receiver takes the best action a^* which maximizes her expected utility (after the normalization by the bounded rationality level β) plus the action-specific random shock, i.e., $a^* = \arg\max_{a\in\mathcal{A}} \beta \cdot v(a \mid \mu) + \varepsilon(a)$. Under the standard assumption that the action-specific random shock $\{\varepsilon(a)\}_{a\in\mathcal{A}}$ is drawn i.i.d. from the Type I extreme value distribution,[24] the probability that action $a \in \mathcal{A}$ is selected over the randomness of $\{\varepsilon(a)\}_{a\in\mathcal{A}}$ is exactly $\exp(\beta \cdot v(a \mid \mu))/(\exp(\beta \cdot v(0 \mid \mu)) + \exp(\beta \cdot v(1 \mid \mu)))$.

Recall $v_i \triangleq v_i(0) - v_i(1)$. Let $\delta \triangleq (\varepsilon(1) - \varepsilon(0))/\beta$. By definition, the cumulative function and density function of random variable δ is $F(\delta) \triangleq \exp(\beta\delta)/(1 + \exp(\beta\delta))$ and $f(\delta) \triangleq \beta\exp(\beta\delta)/(1 + \exp(\beta\delta))^2$, respectively. Given posterior belief μ, the receiver with the action-specific random shock $\{\varepsilon(a)\}_{a\in\mathcal{A}}$ takes action 1 if and only if $\sum_{i\in[m]} \mu_i \cdot (\delta - v_i) \geq 0$. Thus, our problem can be interpreted as the public persuasion problem (Xu, 2020; Dughmi and Xu, 2017) for a continuum population of rational receivers. Specifically, $f(\delta)$ fraction of receiver population is associated with type $\delta \in (-\infty, \infty)$, who has utility $\delta - v_i$ for action 1 and utility zero for action 0 for each state $i \in [m]$.

The linear program $\mathcal{P}_{\texttt{OPT-Primal}}$ can then be interpreted for the aforementioned public persuasion problem. Specifically, variables $\{\pi_i(\delta)\}$ specify a public signaling scheme where each variable $\pi_i(\delta)$ corresponds to the probability that receivers with type greater or equal to δ take action 1 while receivers with type less than δ take action 0. The first (resp. second) constraint in $\mathcal{P}_{\texttt{OPT-Primal}}$ guarantees the persuasiveness (resp. feasibility) of the public signaling scheme.

C Omitted Proofs in Section 2

In this section, we present the omitted proof of Proposition 2.1 in Section 2.

Proposition 2.1. *For every feasible solution $\{\pi_i(\delta)\}$ in program $\mathcal{P}_{\texttt{OPT-Primal}}$, there exists a signaling scheme where for each state $i \in [m]$, the boundedly rational receiver takes action 1 with probability $\int_{-\infty}^{\infty} \pi_i(\delta)W(\delta)\, d\delta$. Furthermore, the sender's optimal expected utility (in the optimal signaling scheme) is equal to the optimal objective value of program $\mathcal{P}_{\texttt{OPT-Primal}}$.*

Proof. Fix an arbitrary feasible solution $\{\pi_i(\delta)\}$ in program $\mathcal{P}_{\texttt{OPT-Primal}}$. We construct a signaling scheme $\pi^\dagger$ as follows.[25] Let the signal space $\Sigma^\dagger \leftarrow \{\delta : \exists i \in [m], \pi_i(\delta) > 0\}$. For each realized state i, let $\pi_i^\dagger(\delta) \leftarrow \pi_i(\delta)$ for each $\delta \in \Sigma^\dagger$. Due to the second constraint in program $\mathcal{P}_{\texttt{OPT-Primal}}$, the constructed signaling scheme $\pi^\dagger$ is valid. When signal $\delta \in \Sigma^\dagger$ is realized, the posterior belief $\mu_i^\dagger(\delta)$ equals $\frac{\lambda_i\pi_i(\delta)}{\sum_{j\in[m]}\lambda_j\pi_j(\delta)}$. Due to the first constraint in program $\mathcal{P}_{\texttt{OPT-Primal}}$, we know that $\sum_{i\in[m]} \mu_i^\dagger(\delta) \cdot (v_i - \delta) = 0$, which implies $\delta = \sum_{i\in[m]} \mu_i^\dagger(\delta) \cdot v_i$. Thus, given realized signal δ, the receiver takes action 1 with probability $W(\delta)$.

So far, we have shown that the sender's optimal expected utility is weakly higher than the optimal objective value of program $\mathcal{P}_{\texttt{OPT-Primal}}$. We finish our proof by converting the optimal signaling scheme π^* (with signal space Σ^*) into a feasible solution $\{\pi_i^\ddagger(\delta)\}$ of program $\mathcal{P}_{\texttt{OPT-Primal}}$ whose objective value equals to the sender's optimal expected utility in π^*.[26] The construction works as follow. First, we initialize $\pi_i^\ddagger(\delta) \leftarrow 0$ for all $i \in [m]$, $\delta \in (-\infty, \infty)$. Next, we enumerate each signal $\sigma \in \Sigma^*$, let $\mu^*(\sigma)$ be the induced posterior belief and $\delta \triangleq \sum_{i\in[m]} \mu_i^*(\sigma)v_i$. By definition, given posterior belief $\mu^*(\sigma)$, the boundedly rational receiver takes action 1 with probability $W(\delta)$. We update $\pi_i^\ddagger(\delta) \leftarrow \pi_i^\ddagger(\delta) + \pi_i^*(\sigma)$ for every state $i \in [m]$. After enumerating every signal $\sigma \in \Sigma^*$, it can be verified that the constructed solution $\pi^\ddagger$ is feasible and its objective value equals the sender's expected utility. ☐

D Omitted Proofs in Section 3

In this section, we present the omitted proofs in Section 3.

[24] The cumulative function of the Type I extreme value distribution is $G(\varepsilon) = \exp(-\exp(-\varepsilon))$.

[25] Here we use the superscript † to denote the constructed signaling scheme.

[26] Here we use the superscript ‡ to denote the constructed feasible solution.

D.1 Interpreting Program $\mathcal{P}_{\texttt{SISU-OPT}}$ for Optimal Censorship of Fully Rational Receiver. The feasibility program $\mathcal{P}_{\texttt{SISU-OPT}}$ recovers the structure of the optimal censorship for a fully rational receiver in SISU environments. For a fully rational receiver (whose bounded rationality level $\beta = \infty$), function $W(\cdot)$ becomes $W(x) = \mathbb{1}[x \le 0]$. In this case, there is no longer a bijection between $\delta^\ddagger \in [0, \infty)$ and $\delta^\dagger \in (-\infty, 0]$ satisfying constraint `dual-feasibility-1`. Instead, the feasible solutions of constraint `dual-feasibility-1` admit one of the two forms: either (i) $\{\delta^\ddagger \in [0, \infty), \delta^\dagger = 0\}$; or (ii) $\{\delta^\ddagger = \infty, \delta^\dagger \in (-\infty, 0]\}$. Note that (i) $\{\delta^\ddagger \in [0, \infty), \delta^\dagger = 0\}$ corresponds to instances where $\sum_{i\in[m]} \lambda_i v_i < 0$ and thus the optimal censorship in Lemma 2.1 selects the threshold state and the threshold state probability such that $\delta^\dagger = 0$, i.e., the fully rational receiver is indifferent between action 0 and action 1 when the pooling signal $\delta^\dagger$ is realized. On the other side, (ii) $\{\delta^\ddagger = \infty, \delta^\dagger \in (-\infty, 0]\}$ corresponds to instances where $\sum_{i\in[m]} \lambda_i v_i \ge 0$ and thus the optimal censorship in Lemma 2.1 sets the threshold state $i^\dagger = \arg\max_i\{v_i : v_i \le \delta^\ddagger\} = m$, i.e., pools all state together and reveals no information.

D.2 Omitted Proof of Proposition 3.1 Below we present the omitted proof of Proposition 3.1. To simplify the analysis, we first introduce the following definition. For any $\delta \in [0, +\infty)$, we define $\kappa(\delta) \in (-\infty, 0]$ such that

$$W'(\kappa(\delta)) = \frac{W(\delta) - W(\kappa(\delta))}{\delta - \kappa(\delta)} \ . \tag{D.1}$$

Clearly, we have $\delta^\dagger = \kappa(\delta^\ddagger)$ where $\delta^\dagger$ and $\delta^\ddagger$ are defined in Theorem 3.1. By the curvature of the function W, namely, W is concave over $(-\infty, 0]$ and convex over $[0, +\infty)$, we have the following property about $\kappa(\cdot)$:

Lemma D.1. *$\kappa(\cdot)$ is a bijection function from $[0, +\infty)$ to $(-\infty, 0]$, i.e, for any $\delta \in [0, +\infty)$, there exists a unique $\kappa(\delta) \in (-\infty, 0]$ that (D.1) holds. Moreover, $\kappa(\cdot)$ is decreasing as $\delta \in [0, +\infty)$ increases.*

Proof. Recall that $W(x) = \frac{1}{1+\exp(\beta x)}$, and $W'(x) = -\beta \cdot \frac{\exp(\beta x)}{(1+\exp(\beta x))^2}$, from (D.1), for a fixed $\delta \ge 0$, $\kappa(\delta)$ is the root of the following function

$$f(x, \delta) \triangleq -\beta \cdot \frac{\exp(\beta x)}{(1 + \exp(\beta x))^2} \cdot (x - \delta) - \frac{1}{1 + \exp(\beta x)} + \frac{1}{1 + \exp(\beta \delta)} \ .$$

Inspecting its first-order partial derivatives, we can see that $\frac{f(x,\delta)}{\partial x} > 0, \forall x < 0$; $\frac{f(x,\delta)}{\partial \delta}\Big|_{x=\kappa(\delta)} > 0$. As a consequence, given $\delta_2 > \delta_1 > 0$, we have $0 = f(\kappa(\delta_1), \delta_1) < f(\kappa(\delta_1), \delta_2)$. From $f(\kappa(\delta_2), \delta_2) = 0$, we know $\kappa(\delta_2) < \kappa(\delta_1)$, which proves the statement. □

With the above definition (D.1), we also define

$$p_i \triangleq -\frac{\sum_{j:j<i} \lambda_j \cdot (v_j - \kappa(v_i))}{\lambda_i \cdot (v_i - \kappa(v_i))} \ . \tag{D.2}$$

Lemma D.2. *For any state $i \in [m]$ with $v_i \ge 0$, if the corresponding $p_i < 0$, then it must have $p_j < 0, \forall j > i$.*

Proof. Consider following two states i_1, i_2 where $i_1 < i_2, v_{i_1} \ge 0$

$$p_{i_1} = -\frac{\sum_{j:j<i_1} \lambda_j(v_j - \kappa(v_{i_1}))}{\lambda_{i_1} \cdot (v_{i_1} - \kappa(v_{i_1}))}, \quad p_{i_2} = -\frac{\sum_{j:j<i_2} \lambda_j(v_j - \kappa(v_{i_2}))}{\lambda_{i_2} \cdot (v_{i_2} - \kappa(v_{i_2}))} \ .$$

Suppose $p_{i_1} < 0$. Since $v_{i_1} - \kappa(v_{i_1}) > 0$, it must imply that $\sum_{j:j<i_1} \lambda_j(v_j - \kappa(v_{i_1})) > 0$. Observe that

$$\sum_{j:j<i_2} \lambda_j(v_j - \kappa(v_{i_2})) = \sum_{j:j<i_1} \lambda_j(v_j - \kappa(v_{i_2})) + \sum_{j:i_1\le j<i_2} \lambda_j(v_j - \kappa(v_{i_2})) > \sum_{j:i_1\le j<i_2} \lambda_j(v_j - \kappa(v_{i_2})) > 0 \ ,$$

where the last inequality follows from the fact that $v_j > 0, \forall i_1 \le j < i_2$ and $\kappa(v_{i_2}) < 0$. Hence, with the fact that $v_{i_2} - \kappa(v_{i_2}) > 0$, one must have $p_{i_2} < 0$. □

We are now ready to prove Proposition 3.1.

PROPOSITION 3.1. *In SISU environments, let π^* (resp. $\hat{\pi}^*$) be the optimal censorship for a boundedly rational receiver with boundedly rational level β (resp. $\hat{\beta}$). If $\beta \le \hat{\beta}$, then the threshold state $i^\dagger$ in π^* is weakly smaller than the threshold state $\hat{i}^\dagger$ in $\hat{\pi}^*$, i.e., $i^\dagger \le \hat{i}^\dagger$; and threshold state probability $p^\dagger \le \hat{p}^\dagger$.*

Proof. We begin the analysis with showing the following observation: Fix a $\delta \in [0, +\infty)$, let $\kappa(\delta)$ (resp. $\hat{\kappa}(\delta)$) be the value that satisfies (D.1) for the bounded rationality level β (resp. $\hat{\beta}$). Then we have

$$\kappa(\delta) \le \hat{\kappa}(\delta) < 0, \quad \text{if } \beta \le \hat{\beta}\ . \tag{D.3}$$

To see this, recall that $W(x) = \frac{1}{1+\exp(\beta x)}$, and $\frac{\partial W(x)}{\partial x} = -\beta \cdot \frac{\exp(\beta x)}{(1+\exp(\beta x))^2}$, from (D.1), $\kappa(\delta)$ is the root of the following function

$$f(x, \beta) \triangleq -\beta \cdot \frac{\exp(\beta x)}{(1+\exp(\beta x))^2} \cdot (x - \delta) - \frac{1}{1+\exp(\beta x)} + \frac{1}{1+\exp(\beta\delta)}\ .$$

Inspecting its first-order partial derivatives, we can see that $\frac{f(x,\beta)}{\partial x} \ge 0, \forall x \le 0$; $\frac{f(x,\beta)}{\partial \beta}\Big|_{x=\kappa(\delta)} \le 0$. As a consequence, given $\hat{\beta} \ge \beta$, we have $f\left(\kappa(\delta), \hat{\beta}\right) \le f\left(\kappa(\delta), \beta\right) = 0$. From $f\left(\hat{\kappa}(\delta), \hat{\beta}\right) = 0$, we know $\hat{\kappa}(\delta) \ge \kappa(\delta)$.

Now given a state i where $v_i \ge 0$, consider the bounded rationality level $\beta, \hat{\beta}$, from (D.3), we have $\kappa(v_i) \le \hat{\kappa}(v_i)$, implying

$$\begin{aligned}
&\sum_{j:j<i} \lambda_j \cdot (v_j - \kappa(v_i)) \ge \sum_{j:j<i} \lambda_j \cdot (v_j - \hat{\kappa}(v_i)) \text{ and } 0 > -\frac{1}{\lambda_i \cdot (v_i - \kappa(v_i))} \ge -\frac{1}{\lambda_i \cdot (v_i - \hat{\kappa}(v_i))};\\
&\Rightarrow\ p_i = -\frac{\sum_{j:j<i} \lambda_j \cdot (v_j - \kappa(v_i))}{\lambda_i \cdot (v_i - \kappa(v_i))} \le -\frac{\sum_{j:j<i} \lambda_j \cdot (v_j - \hat{\kappa}(v_i))}{\lambda_i \cdot (v_i - \hat{\kappa}(v_i))} = \hat{p}_i\ .
\end{aligned} \tag{D.4}$$

The above inequality ensures that the threshold state $\hat{i}^\dagger$ for a larger bounded rationality level $\hat{\beta}$ is no smaller than the threshold state $i^\dagger$ for a smaller bounded rationality level β. If $\hat{i}^\dagger = i^\dagger$, one still has $p^\dagger = \min\{p_{i^\dagger}, 1\} \le \hat{p}^\dagger = \min\{p_{\hat{i}^\dagger}, 1\}$. □

D.3 Omitted Proof of Lemma 3.1 Here we present the omitted proof of Lemma 3.1. With the two properties (see Lemma D.1 and Lemma D.2) for $\kappa(\cdot)$ and p_i we established in Appendix D.2, we prove Lemma 3.1 as follows:

LEMMA 3.1. *There exists a unique feasible solution in program $\mathcal{P}_{\texttt{SISU-OPT}}$.*

Proof. When the set $\{i \in [m] : v_i \ge 0\}$ is not empty, then from the definition (D.1) and the definition (D.2), we know that every feasible solution to the program $\mathcal{P}_{\texttt{SISU-OPT}}$ must be that the threshold state $i^\dagger = \arg\max_{i\in[m]:v_i\ge 0}\{p_i : p_i \ge 0\}$ and $p^\dagger = \min\{p_{i^\dagger}, 1\}$. From Lemma D.2, we know that such threshold state and the pooling probability is unique. On the other hand, if the set $\{i \in [m] : v_i \ge 0\}$ is empty, then Theorem 3.1, together with the definition (D.1) and the definition (D.2), say that $i^\dagger = \arg\max_{i\in[m]:v_i\le 0}\{v_i\}$ and $p^\dagger = 1$, which also guarantees uniqueness of the feasible solution to the program $\mathcal{P}_{\texttt{SISU-OPT}}$. □

D.4 Omitted Proof of Lemma 3.2 Here we present the omitted proof of Lemma 3.2.

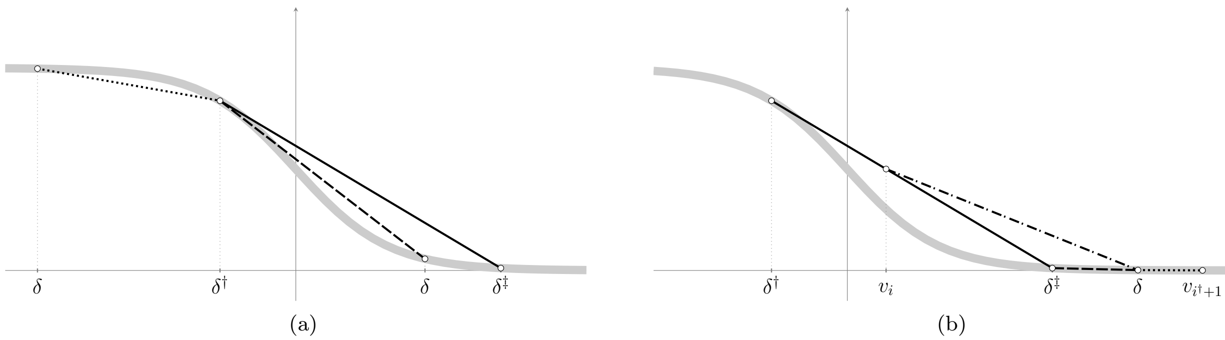


Figure 2: Graphical illustration of the dual assignment feasibility for inequality (3.2) in the proof of Theorem 3.1. The gray solid curve is function $W(\cdot)$. (a): For $\delta \in (-\infty, \delta^\ddagger]$, the dual assignment $\alpha(\delta)$ is the absolute value of the slope of the black solid line. We prove that inequality (3.2) holds by showing that the slope of the black dotted (resp. dashed) line is larger (resp. smaller) than the slope of the black solid line if $\delta \in (-\infty, \delta^\dagger]$ (resp. $\delta \in [\delta^\dagger, \delta^\ddagger]$). (b): For $\delta \in [\delta^\ddagger, v_{i^\dagger+1}]$, the dual assignment $\alpha(\delta)$ is the absolute value of the slope of the black dotted line. We prove that inequality (3.2) holds by showing that the slope of the black dash-dotted line is smaller than the slope of the black dotted line. More specifically, we rewrite inequality (3.2) as inequality (3.5). The right-hand side of inequality (3.5) is the absolute value of the slope of the black dash-dotted line. We lower bound this term by the absolute value of the slope of the black dashed line, which is due to convexity of the function $W(\cdot)$, is larger than the absolute value of the slope of the black dotted line, i.e., the left-hand side of inequality (3.5).

Lemma 3.2. *The constructed dual assignment is a feasible solution to the dual program* $\mathcal{P}_{\texttt{OPT-Dual}}$.

Proof. We first present the analysis for the state $i \in [i^\dagger]$ for the latter two cases:

- Fix an arbitrary $j \in [i^\dagger + 1 : m - 1]$ and an arbitrary $\delta \in [v_j, v_{j+1}]$. Similar to the previous case, after rearranging the terms, inequality (3.2) becomes

$$-\frac{W(\delta) - W(v_{j+1})}{\delta - v_{j+1}} \le -\frac{W(\delta) - \big(W(\delta^\dagger) - W'(\delta^\dagger)(\delta^\dagger - v_i)\big)}{\delta - v_i}$$

  Due to the convexity of function $W(\cdot)$ on $[0, \infty)$,

$$-\frac{W(\delta) - W(v_{j+1})}{\delta - v_{j+1}} \le -\frac{W(\delta) - W(v_{i^\dagger+1})}{\delta - v_{i^\dagger+1}}$$

  and thus the analysis in the previous case can be carried over directly.

- Fix an arbitrary $\delta \in [v_m, \infty)$. By construction $\alpha(\delta) = 0$. Inequality (3.2) becomes

$$W(\delta) \le W(\delta^\dagger) - (\delta^\dagger - v_i)W'(\delta^\dagger) \tag{D.5}$$

  If $v_i \le \delta^\dagger$, inequality (D.5) holds since function $W(\cdot)$ is monotone decreasing. Otherwise, i.e., if $\delta^\dagger \le v_i \le \delta^\ddagger$, inequality (D.5) holds since

$$W(\delta^\dagger) - (\delta^\dagger - v_i)W'(\delta^\dagger) \ge W(\delta^\dagger) - (\delta^\dagger - \delta^\ddagger)W'(\delta^\dagger) \overset{(a)}{=} W(\delta^\ddagger) \overset{(b)}{\ge} W(\delta)$$

  where inequality (a) holds due to constraint `dual-feasibility-1` in program $\mathcal{P}_{\texttt{SISU-OPT}}$; and inequality (b) holds due to the monotonicity of function $W(\cdot)$.

Finally, we verify dual constraints associated with $\pi_i^*(\delta)$ for state $i \in [i^\dagger + 1 : m]$.

- Fix an arbitrary $\delta \in (-\infty, \delta^\ddagger]$. By construction, $\alpha(\delta) = -W(\delta^\dagger)$. By rearranging the terms, inequality (3.3) becomes

$$W'(\delta^\dagger) \le \frac{W(v_i) - W(\delta)}{v_i - \delta}$$

which holds since

$$\frac{W(v_i) - W(\delta)}{v_i - \delta} \overset{(a)}{\ge} \frac{W(\delta^\ddagger) - W(\delta)}{\delta^\ddagger - \delta} \overset{(b)}{\ge} \frac{W(\delta^\ddagger) - W(\delta^\dagger)}{\delta^\ddagger - \delta^\dagger} \overset{(c)}{=} W'(\delta^\dagger)$$

where inequality (a) holds due to the convexity of function $W(\cdot)$ on $[0, \infty)$ and $v_i > \delta^\ddagger$; and inequality (b) and equality (c) hold due to the concavity of function $W(\cdot)$ on $(-\infty, 0]$ and constraint `dual-feasibility-1` in program $\mathcal{P}_{\texttt{SISU-OPT}}$.

- Fix an arbitrary $\delta \in [\delta^\ddagger, v_{i^\dagger+1}]$. By construction, $\alpha(\delta) = -(W(\delta) - W(v_{i^\dagger+1})/(\delta - v_{i^\dagger+1})$. By rearranging the terms, inequality (3.3) becomes

$$\frac{W(v_{i^\dagger+1}) - W(\delta)}{v_{i^\dagger+1} - \delta} \le \frac{W(v_i) - W(\delta)}{v_i - \delta}$$

which holds due to the convexity of function $W(\cdot)$ on $[0, \infty)$ and $\delta \le v_{i^\dagger+1} \le v_i$.

- Fix an arbitrary $j \in [i^\dagger + 1 : m - 1]$ and an arbitrary $\delta \in [v_j, v_{j+1}]$. By construction, $\alpha(\delta) = -(W(\delta) - W(v_{j+1})/(\delta - v_{j+1})$. For state $i \in [i^\dagger + 1 : j]$, by rearranging the terms, inequality (3.3) becomes

$$\frac{W(v_{j+1}) - W(\delta)}{v_{j+1} - \delta} \ge \frac{W(v_i) - W(\delta)}{v_i - \delta}$$

which holds due to the convexity of function $W(\cdot)$ on $[0, \infty)$ and $v_i \le \delta \le v_{j+1}$. Similarly, for state $i \in [j + 1 : m]$, by rearranging the terms, inequality (3.3) becomes

$$\frac{W(v_{j+1}) - W(\delta)}{v_{j+1} - \delta} \le \frac{W(v_i) - W(\delta)}{v_i - \delta}$$

which holds due to the convexity of function $W(\cdot)$ on $[0, \infty)$ and $\delta \le v_{j+1} \le v_i$.

- Fix an arbitrary $\delta \in [v_m, \infty)$. By construction, $\alpha(\delta) = 0$. Here inequality (3.3) holds by the monotonicity of function $W(\cdot)$ straightforwardly. $\square$

$\square$

D.5 Omitted Proof of Theorem 3.2 Here we present the omitted proof of Theorem 3.2.

THEOREM 3.2. *In SISU environments, there exists a problem instance (Example D.1) such that for any direct signaling scheme π, it is $\Omega(m)$-approximation to the optimal signaling scheme.*

EXAMPLE D.1. *Given an arbitrary $m \in \mathbb{N}_+$, consider a problem instance as follows: There are m states. The receiver has bounded rationality level β such that $\beta \ge \exp(m)$. The sender utility $\{u_i\}$, the receiver utility difference $\{v_i\}$, and prior $\{\lambda_i\}$ over state space $[m]$ are*

$$i \in [m]: \qquad u_i = 1, \qquad v_i = i, \qquad \lambda_i = K\left(\exp(\beta i) + 1\right)$$

where $K \triangleq {}^{1}/{(m + \sum_{j\in[m]} \exp(\beta j))}$.

Proof. First, we lowerbound the expected sender utility in the optimal signaling scheme π^* by computing the expected sender utility in the full-information revealing signaling scheme,

$$\textbf{Payoff}[\pi^*] \geq \sum_{i=1} \lambda_i W(v_i) = m \cdot K$$

Next, we upperbound the expected sender utility in the optimal direct signaling scheme $\hat{\pi}$. Suppose the optimal direct signaling scheme $\hat{\pi}$ partitions the state space into $\mathcal{H} \sqcup \{i^\dagger\} \sqcup \mathcal{L}$. Due to the convexity of function $W(\cdot)$ on $[0, \infty)$, the expected sender utility in the optimal direct signaling scheme $\hat{\pi}$ is upperbounded by the expected sender utility in signaling scheme $\tilde{\pi}$ defined as follows,

$$\begin{aligned}
i \in \mathcal{H}: \qquad & \tilde{\pi}_i(\delta) = \mathbb{1}\left[\delta = \frac{\sum_{i\in\mathcal{H}} \lambda_i v_i}{\sum_{i\in\mathcal{H}} \lambda_i}\right] \\
& \tilde{\pi}_{i^\dagger}(\delta) = \mathbb{1}[\delta = v_{i^\dagger}] \\
i \in \mathcal{L}: \qquad & \tilde{\pi}_i(\delta) = \mathbb{1}\left[\delta = \frac{\sum_{i\in\mathcal{L}} \lambda_i v_i}{\sum_{i\in\mathcal{L}} \lambda_i}\right]
\end{aligned}$$

Let $k_h = \max \mathcal{H}$ and $k_l = \max \mathcal{L}$. We have

$$\begin{aligned}
\textbf{Payoff}[\hat{\pi}] &\leq \textbf{Payoff}[\tilde{\pi}] \\
&= \left(\sum_{i\in\mathcal{H}} \lambda_i\right) W\left(\frac{\sum_{i\in\mathcal{H}} \lambda_j v_j}{\sum_{j\in\mathcal{H}} \lambda_j}\right) + \lambda_{i^\dagger} W\left(v_{i^\dagger}\right) + \left(\sum_{i\in\mathcal{L}} \lambda_i\right) W\left(\frac{\sum_{j\in\mathcal{L}} \lambda_j v_j}{\sum_{j\in\mathcal{L}} \lambda_j}\right) \\
&\leq 2\lambda_{k_h} W\left(v_{k_h} - \frac{1}{\beta}\right) + \lambda_{i^\dagger} W\left(v_{i^\dagger}\right) + 2\lambda_{k_l} W\left(v_{k_l} - \frac{1}{\beta}\right) \\
&\leq (4e+1) \cdot K
\end{aligned}$$

where the second inequality holds since $\beta \geq \exp(m)$.

Finally, combining the lower bound (i.e., $m \cdot K$) of $\textbf{Payoff}[\pi^*]$ and the upper bound (i.e., $(4e+1) \cdot K$) of $\textbf{Payoff}[\hat{\pi}]$ finishes the proof. □

E Omitted Proofs in Section 4

E.1 Omitted Proof of Lemma 4.1

Proof. We prove the lemma statement by constructing a feasible signaling scheme π with $\textbf{Payoff}[\pi] = \Theta(K_1 K_2 m)$. In particular, consider the following construction of signaling scheme π:

$$i \in [m-1]: \qquad \pi_i\left(i + \frac{1}{\beta}\right) = 1; \qquad \pi_m\left(i + \frac{1}{\beta}\right) = K_1 \exp(\beta i)$$

It is straightforward to verify by algebra that signaling scheme π constructed above is a feasible solution of program $\mathcal{P}_{\texttt{OPT-Primal}}$.[27] The expected sender utility $\textbf{Payoff}[\pi]$ of signaling scheme π is

$$\begin{aligned}
\textbf{Payoff}[\pi] &= \sum_{i\in[m-1]} \lambda_m u_m \pi_m\left(i + \frac{1}{\beta}\right) W\left(i + \frac{1}{\beta}\right) \\
&= \sum_{i\in[m-1]} K_2 K_1 \exp(\beta i) \frac{1}{1 + \exp(\beta(i + \frac{1}{\beta}))} \\
&= \Theta(K_1 K_2 m) \quad \square
\end{aligned}$$

□

[27] Specifically, in signaling scheme π, each state $i \in [m-1]$ is pooled fully (i.e., $\pi_i(i + 1/\beta) = 1$) on signal $i + 1/\beta$ with the last state m (with probability $\pi_m(1 + 1/\beta) = \exp(\beta i)/K_1$).

E.2 Omitted Proof of Lemma 4.2

LEMMA 4.2. *In Example 4.1, for any state $i \in [m-1]$, the expected sender utility* $\textbf{Payoff}\left[\pi^{(i,\delta)}\right] = O(K_1 K_2)$ *for any* $\delta \in [i, i + {}^{m\log(\beta)}/_{\beta}]$*; and* $\textbf{Payoff}\left[\pi^{(i,\delta)}\right] = o({}^{K_1K_2}/_{m})$ *for any* $\delta \in [i + {}^{m\log(\beta)}/_{\beta}, m]$.

Proof. We first prove the case $\textbf{Payoff}\left[\pi^{(i,\delta)}\right] = O(K_1K_2)$ for any $\delta \in [i, i + {}^{m\log(\beta)}/_{\beta}]$. Recall that in signaling scheme $\pi^{(i,\delta)}$, state i and state m are pooled on signal δ, and all other states are fully revealed, i.e.,

$$\pi_i^{(i,\delta)}(\delta) = 1; \quad \pi_m^{(i,\delta)}(\delta) = \frac{\lambda_i}{\lambda_m}\frac{\delta - i}{m - \delta}; \quad \pi_m^{(i,\delta)}(m) = 1 - \frac{\lambda_i}{\lambda_m}\frac{\delta - i}{m-\delta}; \quad \pi_j^{(i,\delta)}(j) = 1 \forall j \neq i, m\ .$$

The expected sender utility $\textbf{Payoff}\left[\pi^{(i,\delta)}\right]$ of signaling scheme π is

$$\textbf{Payoff}\left[\pi^{(i,\delta)}\right] = \lambda_m u_m \left(\pi_m^{(i,\delta)}(\delta)W(\delta) + \pi_m^{(i,\delta)}(m)W(m)\right)$$

where

$$\begin{aligned}
\lambda_m u_m \pi_m^{(i,\delta)}(m)W(m) &\leq K_2 \frac{1}{1+\exp(\beta m)} \overset{(a)}{=} o(K_1K_2)\\
\lambda_m u_m \pi_m^{(i,\delta)}(\delta)W(\delta) &= K_2K_1\left(m - i - \frac{1}{\beta}\right)\beta\exp(\beta i)\frac{\delta - i}{m-\delta}\frac{1}{1+\exp(\beta\delta)}\\
&= K_1K_2\frac{m - i - \frac{1}{\beta}}{m-\delta}\frac{\exp(\beta i)}{1+\exp(\beta\delta)}\beta(\delta - i) \overset{(b)}{=} O(K_1K_2)
\end{aligned}$$

Here equality (a) holds since ${}^{1}/_{(1+\exp(\beta m))} = o(K_1)$; equality (b) uses two facts that (i) ${}^{(m-i-1/\beta)}/_{(m-\delta)} = O(1)$ since ${}^{\beta}/_{\log(\beta)} \geq 2m$ and thus $\delta \leq i + {}^{m\log(\beta)}/_{\beta} \leq i + {}^{1}/_{2}$; and (ii) $\frac{\exp(\beta i)}{1+\exp(\beta\delta)}\beta(\delta - i) \leq \frac{\beta(\delta-i)}{\exp(\beta(\delta-i))} = O(1)$.

We now prove the case $\textbf{Payoff}\left[\pi^{(i,\delta)}\right] = o({}^{K_1K_2}/_{m})$ for any $\delta \in [i + {}^{m\log(\beta)}/_{\beta}, m]$. It is clear that for every $\delta \geq \delta^{\mathsf{avg}}$, the expected sender utility $\textbf{Payoff}\left[\pi^{(i,\delta)}\right] \leq \textbf{Payoff}\left[\pi^{(i,\delta^{\mathsf{avg}})}\right]$. Thus, it is sufficient to show $\textbf{Payoff}\left[\pi^{(i,\delta)}\right] = o({}^{K_1K_2}/_{m})$ for every $\delta \in [i + {}^{m\log(\beta)}/_{\beta}, \delta^{\mathsf{avg}}]$. By definition,

$$\textbf{Payoff}\left[\pi^{(i,\delta)}\right] = \lambda_m u_m \left(\pi_m^{(i,\delta)}(\delta)W(\delta) + \pi_m^{(i,\delta)}(m)W(m)\right)$$

where

$$\lambda_m u_m \pi_m^{(i,\delta)}(m)W(m) \leq K_2\frac{1}{1+\exp(\beta m)} \overset{(a)}{=} o\left(\frac{K_1K_2}{m}\right)$$

Here equality (a) holds since ${}^{1}/_{(1+\exp(\beta m))} = o({}^{K_1}/_{m})$. It remains to show term $\lambda_m u_m \pi_m^{(i,\delta)}(\delta)W(\delta) = o({}^{K_1K_2}/_{m})$. We show this in two cases based on the value of δ.

- Fix an arbitrary $\delta \in [i + {}^{m\log(\beta)}/_{\beta}, m - {}^{1}/_{2}]$. Note that

$$\begin{aligned}
\lambda_m u_m \pi_m^{(i,\delta)}(\delta)W(\delta) &= K_2K_1\left(m - i - \frac{1}{\beta}\right)\beta\exp(\beta i)\frac{\delta - i}{m-\delta}\frac{1}{1+\exp(\beta\delta)}\\
&\overset{(a)}{\leq} K_1K_2 m\beta\frac{m}{2}\exp(\beta(i-\delta))\\
&\overset{(b)}{\leq} K_1K_2 m\beta\frac{m}{2}\exp\left(\beta\left(i - \left(i + \frac{m\log(\beta)}{\beta}\right)\right)\right)\\
&= o\left(\frac{K_1K_2}{m}\right)
\end{aligned}$$

where inequality (a) holds since $m - i - {}^{1}/_{\beta} \leq m$, ${}^{(\delta - i)}/_{(m-\delta))} \leq {}^{m}/_{2}$; and inequality (b) holds since $\delta \geq i + {}^{m\log(\beta)}/_{\beta}$.

- Fix an arbitrary $\delta \in [m - 1/2, \delta^{\mathsf{avg}}]$. Let $\bar{\delta}^{\mathsf{avg}} \triangleq \frac{\lambda_{m-1}(m-1)+\lambda_m m}{\lambda_{m-1}+\lambda_m}$ be the signal on which state $m-1$ and state m are fully pooled together. It is clear that $\mathbf{Payoff}\left[\pi^{(i,\delta)}\right] \le \mathbf{Payoff}\left[\pi^{(m-1,\bar{\delta}^{\mathsf{avg}})}\right]$ if $\delta \ge \bar{\delta}^{\mathsf{avg}}$. For $\delta \le \bar{\delta}^{\mathsf{avg}}$, note that

$$
\begin{aligned}
\lambda_m u_m \pi_m^{(i,\delta)}(\delta) W(\delta) &= K_2 K_1 \left(m - i - \frac{1}{\beta}\right) \beta \exp(\beta i) \frac{\delta - i}{m - \delta} \frac{1}{1+\exp(\beta\delta)} \\
&\overset{(a)}{\le} K_1 K_2 m \beta \frac{m}{m-\delta} \exp(\beta(m-1-\delta)) \\
&= K_1 K_2 m^2 \beta \frac{\exp(\beta(m-1-\delta))}{m-\delta}
\end{aligned}
$$

where inequality (a) holds since $m - i - 1/\beta \le m$, $\delta - i \le m$ and $\exp(\beta i)/(1+\exp(\beta\delta)) \le \exp(\beta(m-1-\delta))$. Denote function $f(\delta) \triangleq K_1 K_2 m^2 \beta \frac{\exp(\beta(m-1-\delta))}{m-\delta}$. Notably, $f(\bar{\delta}^{\mathsf{avg}})$ also upperbounds $\mathbf{Payoff}\left[\pi^{(m-1,\bar{\delta}^{\mathsf{avg}})}\right]$. Hence, it is sufficient to show $f(\delta) = o(K_1K_2/m)$ for all $\delta \in [m - 1/2, \bar{\delta}^{\mathsf{avg}}]$. Now consider the derivative of function $f(\cdot)$,

$$
\frac{df(\delta)}{d\delta} = K_1 K_2 m^2 \beta \frac{\exp(\beta(m-1-\delta))(1-\beta(m-\delta))}{(m-\delta)^2}
$$

whose sign is determined by the term $1-\beta(m-\delta))$. Since we are considering $\delta \in [m-1/2, \bar{\delta}^{\mathsf{avg}}]$, we conclude the proof by showing that $f(m-1/2) = o(K_1K_2/m)$, and $1-\beta(m-\bar{\delta}^{\mathsf{avg}}) \le 0$. By definition,

$$
f\left(m - \frac{1}{2}\right) = K_1 K_2 m^2 \beta \frac{\exp\left(\beta\left(m-1-\left(m-\frac{1}{2}\right)\right)\right)}{m - (m-\frac{1}{2})} = o\left(\frac{K_1K_2}{m}\right)
$$

and

$$
\begin{aligned}
\bar{\delta}^{\mathsf{avg}} &= \frac{\lambda_{m-1}(m-1)+\lambda_m m}{\lambda_{m-1}+\lambda_m} \\
&= \frac{K_1K_2\left(1-\frac{1}{\beta}\right)\beta\exp(\beta(m-1))(m-1) + K_2 m}{K_1K_2\left(1-\frac{1}{\beta}\right)\beta\exp(\beta(m-1)) + K_2} \\
&= \frac{\frac{1}{\sum_{j\in[m-1]}\exp(\beta j)}\left(1-\frac{1}{\beta}\right)\beta\exp(\beta(m-1))(m-1)+m}{\frac{1}{\sum_{j\in[m-1]}\exp(\beta j)}\left(1-\frac{1}{\beta}\right)\beta\exp(\beta(m-1))+1} \\
&\overset{(a)}{\le} \frac{\frac{1}{2}\left(1-\frac{1}{\beta}\right)(m-1)+m}{\frac{1}{2}\left(1-\frac{1}{\beta}\right)+1} \le m - \frac{1}{3} \overset{(b)}{\le} m - \frac{1}{\beta}
\end{aligned}
$$

where inequalities (a) and (b) hold for every $m \ge 3$. □

□

E.3 Omitted Proof of Theorem 4.2

Remaining Proof of Theorem 4.2. We now prove that there always exists a direct signaling scheme that is $O(m)$-approximation. Similarly, let $\pi^\dagger$ with signal space $\Sigma^\dagger$ be the signaling scheme stated in Lemma 4.3. Suppose the signal $\sigma_{ij} \in \Sigma^\dagger$ is induced from the pair of state (i,j). We use $\widetilde{U}_{ij}$ denote the expected sender utility induced from the signal σ_{ij}, i.e., $\widetilde{U}_{ij} \triangleq (\lambda_i u_i \pi_i^\dagger(\sigma_{ij}) + \lambda_j u_j \pi_j^\dagger(\sigma_{ij}))W(\sigma_{ij})$. Let $(\widetilde{i},\widetilde{j}) \triangleq \arg\max_{(i,j)} \widetilde{U}_{ij}$. Then, together with the properties (i)-(ii) and $|\Sigma^\dagger| \le 2m$, we know that $\mathbf{Payoff}\left[\pi^\dagger\right] \le 2m \cdot \widetilde{U}_{\widetilde{i}\widetilde{j}}$. Now consider the following direct signaling scheme $\widetilde{\pi}$:

$$
\begin{aligned}
&\widetilde{\pi}_{\widetilde{i}}(\delta) = \pi_i^{\dagger}(\sigma_{\widetilde{i}\widetilde{j}})\mathbb{1}\left[\delta = \sigma_{\widetilde{i}\widetilde{j}}\right], \quad \widetilde{\pi}_{\widetilde{i}}(\delta) = \left(1 - \pi_i^{\dagger}(\sigma_{\widetilde{i}\widetilde{j}})\right)\mathbb{1}\left[\delta = \widetilde{\delta}\right]; \\
&\widetilde{\pi}_{\widetilde{j}}(\delta) = \pi_j^{\dagger}(\sigma_{\widetilde{i}\widetilde{j}})\mathbb{1}\left[\delta = \sigma_{\widetilde{i}\widetilde{j}}\right], \quad \widetilde{\pi}_{\widetilde{j}}(\delta) = \left(1 - \pi_j^{\dagger}(\sigma_{\widetilde{i}\widetilde{j}})\right)\mathbb{1}\left[\delta = \widetilde{\delta}\right]; \\
i \in [m] \setminus \{\widetilde{i}, \widetilde{j}\}, \quad &\widetilde{\pi}_i(\delta) = \mathbb{1}\left[\delta = \widetilde{\delta}\right]
\end{aligned}
$$

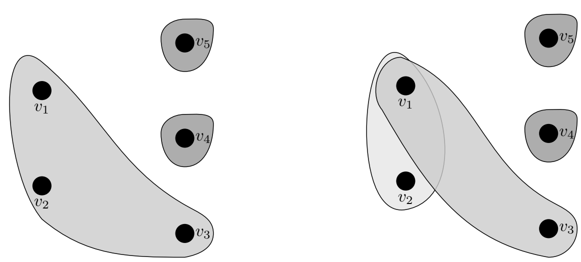
where $\widetilde{\delta} \triangleq \frac{\lambda_{\widetilde{i}}(1-\pi_i^{\dagger}(\sigma_{\widetilde{i}\widetilde{j}}))v_{\widetilde{i}} + \lambda_{\widetilde{j}}(1-\pi_j^{\dagger}(\sigma_{\widetilde{i}\widetilde{j}}))v_{\widetilde{j}} + \sum_{i\in[m]\setminus\{\widetilde{i},\widetilde{j}\}} \lambda_i v_i}{\lambda_{\widetilde{i}}(1-\pi_i^{\dagger}(\sigma_{\widetilde{i}\widetilde{j}})) + \lambda_{\widetilde{j}}(1-\pi_j^{\dagger}(\sigma_{\widetilde{i}\widetilde{j}})) + \sum_{i\in[m]\setminus\{\widetilde{i},\widetilde{j}\}} \lambda_i}$. Essentially, direct signaling scheme $\widetilde{\pi}$ has the same signaling structure as the signaling scheme $\pi^{\dagger}$ on inducing the signal $\sigma_{\widetilde{i}\widetilde{j}}$, and then pools all remaining states at the same signal $\widetilde{\delta}$. By construction, it is easy to verify that $\textbf{Payoff}[\widetilde{\pi}] \geq \widetilde{U}_{\widetilde{i}\widetilde{j}}$, which gives an $O(m)$-approximation of the signaling scheme $\widetilde{\pi}$. $\square$

E.4 Omitted Proof of Lemma 4.3 We start with the first step – a characterization of an optimal signaling scheme that has the same two properties as in Lemma 4.3.

Step 1- a characterization of the structure of an optimal signaling scheme.

Lemma E.1. *In SDSU environments, for a boundedly rational receiver, there exists an optimal signaling scheme π^* using at most $m(m+1)/2$ signals and this optimal signaling scheme π^* satisfies the two properties (i) (ii) in Lemma 4.3.*

We provide a graphical illustration for the structure of optimal signaling schemes characterized in the above Lemma E.1.



(a) (b)

Figure 3: Structure graphical illustration for optimal signaling schemes characterized in Theorem 3.1 (Figure 3a) and Lemma E.1 (Figure 3b) in a SISU environment. Each state is the black dot, and all the states in a gray shaded region imply that there exists a signal induced from these states. In both figures, the receiver is fully rational, i.e., $\beta = \infty$. The SISU environment is specified as below: $v_1 = -1.5, v_2 = 0.5, v_3 = 1, v_4 = 1.5, v_5 = 2$, and $u_i = 1, \lambda_i = 0.2, \forall i \in [5]$. For this problem instance, it can be shown that a censorship signaling scheme $\widetilde{\pi}^*$ (Figure 3a) is optimal: $\widetilde{\Sigma}^* = \{\sigma_1, \sigma_2, \sigma_3\}$, $\widetilde{\pi}_i^*(\sigma_1) = 1, \forall i \in [3]; \widetilde{\pi}_4^*(\sigma_2) = 1; \widetilde{\pi}_5^*(\sigma_3) = 1$. Meanwhile, a signaling scheme π^* (Figure 3b) that satisfies the two properties in Lemma E.1 is also optimal: $\Sigma^* = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, $\pi_1^*(\sigma_1) = 1/3, \pi_2^*(\sigma_1) = 1; \pi_1^*(\sigma_2) = 2/3, \pi_3^*(\sigma_2) = 1; \pi_4^*(\sigma_3) = 1; \pi_5^*(\sigma_4) = 1$.

Remark E.1. *We would like to note that an application of the Caratheodory's theorem shows that m signals are sufficient for optimal signaling scheme. However, such characterization does not shed much light on the structure of optimal signaling scheme. To prove Lemma 4.3, we resort to characterizing an optimal signaling scheme that uses more signals but has more structural properties that we can leverage to study censorship/direct signaling schemes.*

Proof overview of Lemma E.1. At a high level, the proof of Lemma E.1 proceeds in two steps. In step 1a, we present a reduction from arbitrary signaling schemes to signaling schemes that satisfy property (i). Specifically, given an arbitrary signaling scheme π, we can construct a new signaling scheme $\pi^{\dagger}$ which satisfies property (i) and achieves the same expected sender utility as the original signaling scheme.[28] In step 1b, we provide an approach to convert any signaling scheme $\pi^\dagger$ which satisfies property (i) to a new signaling scheme $\pi^\ddagger$ which satisfies both properties (i) (ii), and achieves weakly higher expected sender utility. Informally, given a signaling scheme $\pi^\dagger$ from the first step, we can obtain the signaling scheme $\pi^\ddagger$ by optimizing the pooling structure for each pair of states while holding signals from all other pairs fixed. Loosely speaking, this reduces our task to identify optimal signaling scheme when the state space is binary. Hence, we introduce a technical lemma (Lemma E.5), showing that the optimal signaling schemes are censorship signaling schemes when the sate space is binary, which may be of independent interest.

Below we provide detailed discussion and related lemmas for the above mentioned two steps. In the end of this subsection, we combine all pieces together to conclude the proof of Lemma E.1.

Step 1a- reduction to signaling schemes with property (i). In this step, we argue that it is without loss of generality to consider signaling schemes that satisfy property (i) in Lemma E.1.

Lemma E.2. *In SDSU environments, for a boundedly rational receiver, for an arbitrary signaling scheme π, there exists a signaling scheme $\pi^\dagger$ with signal space $\Sigma^\dagger$ such that*

- *each signal $\sigma \in \Sigma^\dagger$ is induced by at most two states,*
- *signaling scheme $\pi^\dagger$ achieves the same expected sender utility as signaling scheme π.*

Informally, we can construct the signaling scheme $\pi^\dagger$ in Lemma E.2 as follows. For each signal σ in the original signaling scheme π, we decompose it into multiple signals, each of which is induced by at most two states, and satisfies some other requirements. The feasibility of this decomposition is guaranteed by the following lemma.

Lemma E.3. (Feng et al., 2022) *Let X be a random variable with discrete support $\mathsf{supp}(X)$. There exists a positive integer K, a finite set of K random variables $\{X_k\}_{k\in[K]}$, and convex combination coefficients $\mathbf{f} \in [0,1]^K$ with $\sum_{k\in[K]} f_k = 1$ such that:*

(i) Bayesian-plausibility: for each $k \in [K]$, $\mathbb{E}[X_k] = \mathbb{E}[X]$;

(ii) Binary-support: for each $k \in [K]$, the size of X_k's support is at most 2, i.e., $|\mathsf{supp}(X_k)| \le 2$

(iii) Consistency: for each $x \in \mathsf{supp}(X)$, $Pr[X = x] = \sum_{k\in[K]} f_k \cdot Pr[X_k = x]$

Proof of Lemma E.2. Fix an arbitrary signaling scheme π with signal space Σ. Recall that $\pi_i(\sigma)$ is the probability mass (or density) that signal σ is realized when the realized state is state i.

Now we describe the construction of $\pi^\dagger$ and its signal space $\Sigma^\dagger$. Initially, we set $\Sigma^\dagger \leftarrow \emptyset$. For each signal $\sigma \in \Sigma$, let $\mu_i(\sigma) \triangleq \frac{\lambda_i \pi_i(\sigma)}{\sum_{j\in[m]} \lambda_j \pi_j(\sigma)}$ be its induced posterior belief for each state i. Consider the following random variable X where $\Pr[X = v_i] = \mu_i(\sigma)$ for each $i \in [m]$. Let integer K, random variables $\{X_k\}_{k\in[K]}$, and convex combination coefficients $\mathbf{f} \in [0,1]^K$ be the elements in Lemma E.3 for the aforementioned random variable X. Add K signals $\{\sigma^{(1)}, \dots, \sigma^{(K)}\}$ into the signal space $\Sigma^\dagger$, i.e., $\Sigma^\dagger \leftarrow \Sigma^\dagger \cup \{\sigma^{(1)}, \dots, \sigma^{(K)}\}$. For each $k \in [K]$, set $\pi^\dagger_i(\sigma^{(k)}) \leftarrow \frac{1}{\lambda_i} f_k \cdot \Pr[X_k = v_i] \cdot (\sum_{j\in[m]} \lambda_j \pi_j(\sigma))$. Note that this construction ensures that

$$\sum_{k\in[K]} \pi^\dagger_i(\sigma^{(k)}) = \sum_{k\in[K]} \frac{1}{\lambda_i} f_k \cdot \Pr[X_k = v_i] \cdot \left(\sum_{j\in[m]} \lambda_j \pi_j(\sigma)\right) \overset{(a)}{=} \frac{1}{\lambda_i} \Pr[X = v_i] \cdot \left(\sum_{j\in[m]} \lambda_j \pi_j(\sigma)\right) = \pi_i(\sigma)$$

where equality (a) holds due to the "consistency" property in Lemma E.3. Hence, the constructed signaling scheme $\pi^\dagger$ is feasible.

Additionally, the "binary-support" property in Lemma E.3 ensures that signaling scheme $\pi^\dagger$ satisfies that each signal from $\Sigma^\dagger$ is induced by at most two states.

Finally, to see that signaling scheme $\pi^\dagger$ achieves the same expected sender utility as signaling scheme π, consider the following coupling between these two signaling schemes: whenever signal $\sigma \in \Sigma$ is realized in signaling

[28]In the remaining of this subsection, we use superscript † to denote the constructed signaling schemes satisfying property (i), and superscript ‡ to denote the constructed signaling schemes satisfying properties (i) (ii).

scheme π, sample the corresponding signal $\sigma^{(k)}$ with probability f_k for each $k \in [K]$. This coupling is well-defined due to the "consistency" property in Lemma E.3. Invoking the "Bayesian-plausibility" property in Lemma E.3, from the receiver's perspective, her expected utility given the posterior belief $\mu(\sigma^{(k)})$ under signaling scheme $\pi^\dagger$ is the same as her expected utility given the posterior belief $\mu(\sigma)$ under signaling scheme π. Thus the probabilities that the receiver takes action 1 are the same in both signaling schemes, yielding the same expected utility to the sender. $\square$

Step 1b- reduction to signaling schemes with properties (i) and (ii). In this step, we argue that it is without loss of generality to consider signaling schemes which satisfy properties (i) and (ii) in Lemma E.1.

LEMMA E.4. *In SDSU environments, for a boundedly rational receiver, given any signaling scheme $\pi^\dagger$ with signal space $\Sigma^\dagger$ where each signal $\sigma \in \Sigma^\dagger$ is induced by at most two states, there exists a signaling scheme $\pi^\ddagger$ with signal space $\Sigma^\ddagger$ such that*

- *each signal $\sigma \in \Sigma^\ddagger$ is induced by at most two states,*
- *each pair of states is pooled at most one signal,*
- *signaling scheme $\pi^\ddagger$ achieves weakly higher expected sender utility as signaling scheme $\pi^\dagger$.*

Proof. Fix an arbitrary signaling scheme $\pi^\dagger$ with signal space $\Sigma^\dagger$ where each signal is induced by at most two states. Below we describe the construction of $\pi^\ddagger$ and its signal space $\Sigma^\ddagger$. Initially, we set $\Sigma^\ddagger \leftarrow \emptyset$.

For each pair of states (i, j), let $\Sigma^\dagger_{ij} \subseteq \Sigma^\dagger$ be the subset of signals, each of which is induced by state i and state j, i.e., $\Sigma^\dagger_{ij} \triangleq \{\sigma \in \Sigma^\dagger : \mathsf{supp}(\mu(\sigma)) = \{i, j\}\}$. For ease of presentation, we introduce auxiliary notations $p^{(1)}_{ij}$ (resp. $p^{(2)}_{ij}$) to denote the probability that the realized state is i (resp. j), and the realized signal is from $\Sigma^\dagger_{ij}$, i.e., $p^{(1)}_{ij} \triangleq \int_{\sigma\in\Sigma^\dagger_{ij}} \lambda_i \pi^\dagger_i(\sigma)\, d\sigma$ and $p^{(2)}_{ij} \triangleq \int_{\sigma\in\Sigma^\dagger_{ij}} \lambda_j \pi^\dagger_j(\sigma)\, d\sigma$. Consider the program $\mathcal{P}_{\texttt{OPT-Primal}}$ on the following binary-state instance $\mathcal{I}_{ij} = (\hat{m}, \{\hat{\lambda}_k\}, \{\hat{v}_k\}, \{\hat{u}_k\})$:

$$\hat{m} \leftarrow 2, \qquad \hat{v}_1 \leftarrow v_i, \qquad \hat{v}_2 \leftarrow v_j, \qquad \hat{u}_1 \leftarrow u_i, \qquad \hat{u}_2 \leftarrow u_j,$$
$$\hat{\lambda}_1 \leftarrow \frac{p^{(1)}_{ij}}{p^{(1)}_{ij} + p^{(2)}_{ij}}, \qquad \hat{\lambda}_2 \leftarrow \frac{p^{(1)}_{ij}}{p^{(1)}_{ij} + p^{(2)}_{ij}}$$

Notably, $\left\{ \frac{\lambda_i \pi^\dagger_i(\sigma)}{p^{(1)}_{ij}} \cdot \mathbb{1}\left[\sigma \in \Sigma^\dagger_{ij}\right], \frac{\lambda_j \pi^\dagger_j(\sigma)}{p^{(2)}_{ij}} \cdot \mathbb{1}\left[\sigma \in \Sigma^\dagger_{ij}\right] \right\}$ is a feasible solution of program $\mathcal{P}_{\texttt{OPT-Primal}}$ on the binary-state instance $\mathcal{I}_{ij}$. Now, let $\{\hat{\pi}^*_1(\sigma), \hat{\pi}^*_2(\sigma)\}$ with signal space $\hat{\Sigma}^*_{ij}$ be the optimal solution of program $\mathcal{P}_{\texttt{OPT-Primal}}$ on the binary-state instance $\mathcal{I}_{ij}$. We add signals from $\hat{\Sigma}^*_{ij}$ into the signal space $\Sigma^\ddagger$, i.e., $\Sigma^\ddagger \leftarrow \Sigma^\ddagger \cup \hat{\Sigma}^*_{ij}$, and set $\pi^\ddagger_i(\sigma) \leftarrow \frac{p^{(1)}_{ij}}{\lambda_i} \cdot \hat{\pi}^*_1(\sigma)$, $\pi^\ddagger_j(\sigma) \leftarrow \frac{p^{(2)}_{ij}}{\lambda_j} \cdot \hat{\pi}^*_2(\sigma)$ for each signal $\sigma \in \hat{\Sigma}^*_{ij}$. It is straightforward to verify that the constructed signaling scheme $\pi^\ddagger$ with signal space $\Sigma^\ddagger$ is feasible, and each signal $\sigma \in \Sigma^\dagger$ is induced by at most two states by construction.

Now we argue that each pair of states in signaling scheme $\pi^\ddagger$ is pooled at most one signal. By our construction of signaling scheme $\pi^\ddagger$, it is sufficient to show that for each pair of states (i, j), the optimal solution $\{\hat{\pi}^*_i(\sigma), \hat{\pi}^*_j(\sigma)\}$ in program $\mathcal{P}_{\texttt{OPT-Primal}}$ on binary-state instance $\mathcal{I}_{ij}$ is a censorship signaling scheme (and thus it is pooled at most one signal). We prove this statement by leveraging the following lemma (Lemma E.5) that characterizes the optimal signaling scheme of any binary state instance is indeed a censorship signaling scheme. The proof, deferred to Appendix E, is based on a primal-dual analysis similar to the one for Theorem 3.1.

LEMMA E.5. *In SDSU environments with binary state space (i.e., $m = 2$), there exists a censorship signaling scheme that is an optimal signaling scheme.*

The proof of Lemma E.5 follows similar primal-dual analysis of the one for Theorem 3.1, we thus defer to proof to Appendix E.5.

Finally, we verify that expected sender utility $\textbf{Payoff}[\pi^\ddagger]$ is weakly higher than the expected sender utility $\textbf{Payoff}[\pi^\dagger]$. Note that

$$\begin{aligned}
\textbf{Payoff}[\pi^\dagger] &\overset{(a)}{=} \sum_{(i,j)} \int_{\sigma\in\Sigma_{ij}^\dagger} \left(\lambda_i u_i \pi_i^\dagger(\sigma) + \lambda_j u_j \pi_j^\dagger(\sigma)\right) W(\sigma)\, d\sigma \\
&= \sum_{(i,j)} \left(p_{ij}^{(1)} + p_{ij}^{(2)}\right) \int_{\sigma\in\Sigma_{ij}^\dagger} \left(\frac{p_{ij}^{(1)}}{p_{ij}^{(1)}+p_{ij}^{(2)}} u_i \frac{\lambda_i \pi_i^\dagger(\sigma)}{p_{ij}^{(1)}} + \frac{p_{ij}^{(2)}}{p_{ij}^{(1)}+p_{ij}^{(2)}} u_i \frac{\lambda_j \pi_i^\dagger(\sigma)}{p_{ij}^{(2)}}\right) W(\sigma)\, d\sigma \\
&\overset{(b)}{\le} \sum_{(i,j)} \left(p_{ij}^{(1)} + p_{ij}^{(2)}\right) \int_{\sigma\in\Sigma_{ij}^\dagger} \left(\frac{p_{ij}^{(1)}}{p_{ij}^{(1)}+p_{ij}^{(2)}} u_i \frac{\lambda_i \pi_i^\ddagger(\sigma)}{p_{ij}^{(1)}} + \frac{p_{ij}^{(2)}}{p_{ij}^{(1)}+p_{ij}^{(2)}} u_i \frac{\lambda_j \pi_i^\ddagger(\sigma)}{p_{ij}^{(2)}}\right) W(\sigma)\, d\sigma \\
&= \sum_{(i,j)} \int_{\sigma\in\Sigma_{ij}^\dagger} \left(\lambda_i u_i \pi_i^\ddagger(\sigma) + \lambda_j u_j \pi_j^\ddagger(\sigma)\right) W(\sigma)\, d\sigma \\
&\overset{(c)}{=} \textbf{Payoff}[\pi^\ddagger]
\end{aligned}$$

where equalities (a) (c) use Proposition 2.1. To see why inequality (b) holds, note that the left-hand side of inequality (b) is the objective value of solution $\left\{\frac{\lambda_i\pi_i^\dagger(\sigma)}{p_{ij}^{(1)}}\cdot \mathbb{1}\left[\sigma\in\Sigma_{ij}^\dagger\right], \frac{\lambda_j\pi_j^\dagger(\sigma)}{p_{ij}^{(2)}}\cdot \mathbb{1}\left[\sigma\in\Sigma_{ij}^\dagger\right]\right\}$ in program $\mathcal{P}_{\texttt{OPT-Primal}}$ on binary-state instance $\mathcal{I}_{ij}$, while the right-hand side of inequality (b), by the construction of $\pi^\ddagger$, is the optimal objective value in this program. □

Now we are ready to prove Lemma E.1.

Proof of Lemma E.1. Invoking Lemma E.2 and Lemma E.4, we know that there exists an optimal signaling scheme where (i) each signal is induced by at most two states, and (ii) each pair of states pools on at most one signal. Note that property (i) and property (ii) together imply that its signal space has $\frac{m(m+1)}{2}$ signals. □

In below, we provide the analysis of the second step for the proof of Lemma 4.3.

Step 2- a connection to fractional generalized assignment problem. Due to properties (i) and (ii) of the optimal signaling scheme π^* stated in Lemma E.1, there is at most one signal realized by each pair of states (i,j), i.e., $|\{\sigma : \pi_i^*(\sigma) > 0 \wedge \pi_j^*(\sigma) > 0\}| \le 1$. For ease of presentation, we assume $|\{\sigma : \pi_i^*(\sigma) > 0 \wedge \pi_j^*(\sigma) > 0\}| = 1$ for each pair (i,j), and denote it as σ_{ij}.[29] Furthermore, we define set of pairs $E \triangleq \{(i,j) : \pi_i^*(\sigma_{ij}) \ge \pi_j^*(\sigma_{ij})\}$. Note that the expected sender utility $\textbf{Payoff}[\pi^*] = \sum_{(i,j)\in E}(\lambda_i u_i \pi_i^*(\sigma_{ij}) + \lambda_j u_j \pi_j^*(\sigma_{ij}))W(\sigma_{ij})$ can be upper bounded by the optimal value of the following linear program,

$$(\mathcal{P}_{\texttt{SDSU-OPT}}) \qquad \begin{aligned}
\max_{\mathbf{x}\ge\mathbf{0}} \quad & \sum_{(i,j)\in E} \left(\lambda_i u_i + \lambda_j u_j \frac{\pi_j^*(\sigma_{ij})}{\pi_i^*(\sigma_{ij})}\right) W(\sigma_{ij}) x_{ij} \quad \text{s.t.} \\
& \sum_{j:(i,j)\in E} x_{ij} \le 1 \qquad\qquad i\in[m] \\
& \sum_{i:(i,j)\in E} \frac{\pi_j^*(\sigma_{ij})}{\pi_i^*(\sigma_{ij})} \cdot x_{ij} \le 1 \qquad\qquad j\in[m]
\end{aligned}$$

LEMMA E.6. *The expected sender utility* $\textbf{Payoff}[\pi^*]$ *of the optimal signaling scheme* π^* *is at most the optimal objective value of program* $\mathcal{P}_{\texttt{SDSU-OPT}}$.

Proof. Consider the following assignment $\mathbf{x}$ of program $\mathcal{P}_{\texttt{SDSU-OPT}}$,

$$i\in[m],\quad j\in[m]: \qquad x_{ij} \leftarrow \pi_i^*(\sigma_{ij})$$

[29] The analysis in this subsection extends trivially if $|\{\sigma : \pi_i^*(\sigma) > 0 \wedge \pi_j^*(\sigma) > 0\}| = 0$ for some pair (i,j).

By construction, the objective value of the constructed assignment equals $\mathbf{Payoff}[\pi^*]$. Now, we show the feasibility of the constructed assignment. Note the feasibility of optimal signaling scheme π^* implies that for each state $i \in [m]$, $\sum_{j\in[m]} \pi_i^*(\sigma_{ij}) = 1$. Thus,

$$\sum_{j\in[m]} x_{ij} = \sum_{j\in[m]} \pi_i^*(\sigma_{ij}) \le 1; \quad \sum_{i\in[m]} \frac{\pi_j^*(\sigma_{ij})}{\pi_i^*(\sigma_{ij})} \cdot x_{ij} = \sum_{i\in[m]} \frac{\pi_j^*(\sigma_{ij})}{\pi_i^*(\sigma_{ij})} \cdot \pi_i^*(\sigma_{ij}) = \sum_{i\in[m]} \pi_j^*(\sigma_{ij}) \le 1$$

which finishes the proof. □

We remark that the program $\mathcal{P}_{\texttt{SDSU-OPT}}$ has the same formulation as the fractional generalized assignment problem: there are m items and m bins. Each bin has a unit budget. Each pair of item i and bin j such that $(i,j) \in E$ has value $(\lambda_i u_i + {}^{\lambda_j u_j \pi_j^*(\sigma_{ij})}/_{\pi_i^*(\sigma_{ij})})W(\sigma_{ij})$ and cost ${}^{\pi_j^*(\sigma_{ij})}/_{\pi_i^*(\sigma_{ij})}$. With this connection to the generalized assignment problem, we use the following established result about the optimal integral solution of program $\mathcal{P}_{\texttt{SDSU-OPT}}$.

LEMMA E.7. (THEOREM 2.1 AND ITS PROOF IN SHMOYS AND TARDOS, 1993) *The optimal integral solution of program $\mathcal{P}_{\texttt{SDSU-OPT}}$ is a 2-approximation to the optimal fraction solution of program $\mathcal{P}_{\texttt{SDSU-OPT}}$.*

Now we are ready to prove Lemma 4.3.

Proof of Lemma 4.3. Let $\mathbf{x}^\dagger$ be the optimal integral solution of program $\mathcal{P}_{\texttt{SDSU-OPT}}$. Consider a signaling scheme $\pi^\dagger$ constructed as follows. First, initialize the signal space $\Sigma^\dagger \leftarrow \emptyset$. Second, for each pair of state $(i,j) \in E$, if $x_{ij}^\dagger > 0$, update $\Sigma^\dagger \leftarrow \Sigma^\dagger \cup \{\sigma_{ij}\}$, $\pi_i^\dagger(\sigma_{ij}) \leftarrow {}^{x_{ij}^\dagger}/_2$, and $\pi_j^\dagger(\sigma_{ij}) \leftarrow {}^{(x_{ij}^\dagger \pi_j^*(\sigma_{ij}))}/_{(2\pi_i^*(\sigma_{ij}))}$. Third, for each state $i \in [m]$, if $\sum_{\sigma\in\Sigma^\dagger} \pi_i^\dagger(\sigma) < 1$, update $\Sigma^\dagger \leftarrow \Sigma^\dagger \cup \{v_i\}$, $\pi_i^\dagger(v_i) \leftarrow 1 - \sum_{\sigma\in\Sigma^\dagger} \pi_i^\dagger(\sigma)$.

Now we verify that the constructed signaling scheme $\pi^\dagger$ is feasible, i.e., for each state $i \in [m]$, $\sum_{\sigma\in\Sigma^\dagger} \pi_i^\dagger(\sigma) = 1$. By construction, the feasibility is guaranteed since that for each state $i \in [m]$,

$$\sum_{j:(i,j)\in E} \pi_i^\dagger(\sigma_{ij}) + \sum_{j:(j,i)\in E} \pi_i^\dagger(\sigma_{ji}) = \sum_{j:(i,j)\in E} \frac{1}{2} x_{ij}^\dagger + \sum_{j:(j,i)\in E} \frac{1}{2} \frac{\pi_i^*(\sigma_{ji})}{\pi_j^*(\sigma_{ji})} x_{ji}^\dagger \le \frac{1}{2} + \frac{1}{2} = 1$$

where the inequality holds due to the feasibility of solution $\mathbf{x}^\dagger$.

Next, we verify that the constructed signaling scheme $\pi^\dagger$ satisfies properties stated in Lemma 4.3. Note the two properties same as in Lemma E.1 are guaranteed by construction straightforwardly. By construction, the expected sender utility $\mathbf{Payoff}[\pi^\dagger]$ is a 2-approximation to the objective value of the optimal integral solution $\mathbf{x}^\dagger$. Invoking Lemma E.6 and Lemma E.7, we conclude that signaling scheme $\pi^\dagger$ is 4-approximation to the optimal signaling scheme.

Finally, since the optimal integral solution $\mathbf{x}^\dagger$ has at most m non-zero entries, i.e., $|\{x_{ij}^\dagger : x_{ij}^\dagger > 0\}| \le m$, the constructed signal space $\Sigma^\dagger$ has at most $|\{x_{ij}^\dagger : x_{ij}^\dagger > 0\}| + m \le 2m$ signals. □

E.5 Proof of Lemma E.5 We now present a more detailed statement for Lemma E.5 and then present its associated proof.

LEMMA E.8. *In SDSU environments with binary state space (i.e., $m = 2$), there exists an optimal signaling scheme π^* for a boundedly rational receiver that is a censorship signaling scheme. In particular, define $\gamma(\delta) \triangleq \frac{v_1-\delta}{v_2-\delta} + \frac{W(v_2)-W(\delta)}{v_2-\delta} \cdot \frac{1}{W'(\delta)} \cdot \left(1 - \frac{v_1-\delta}{v_2-\delta}\right)$. Let $\widehat{\delta}$ satisfy $\gamma(\widehat{\delta}) = {}^{u_1}/_{u_2}$, and define $\delta^\dagger \triangleq \min\left\{\max\left\{v_1, \widehat{\delta}\right\}, \lambda_1 v_1 + \lambda_2 v_2\right\}$; $p^\dagger \triangleq \frac{\lambda_1(\delta^\dagger - v_1)}{\lambda_2(v_2-\delta^\dagger)}$. Then the optimal signaling π^* is*

$$\pi_1^*\left(\delta^\dagger\right) = 1;\ \pi_2^*\left(\delta^\dagger\right) = p^\dagger,\ \pi_2^*\left(v_2\right) = 1 - p^\dagger\ . \tag{E.6}$$

A few useful observations of the above result are as follows. First, by inspecting the first-order derivative, we know that the function $\gamma(\cdot)$ is monotone decreasing. Second, we always have $\delta^\dagger \in [v_1, \lambda_1 v_1 + \lambda_2 v_2]$ and thus $p^\dagger \in [0,1]$. Third, (a) when $\widehat{\delta} \le v_1$, we have $\delta^\dagger = v_1$ and $p^\dagger = 0$, and thus full information revealing is optimal; (b) when $v_1 < \widehat{\delta} < \lambda_1 v_1 + \lambda_2 v_2$, we have $\delta^\dagger = \widehat{\delta}$ and $p^\dagger \in (0,1)$, and thus partial information revealing is optimal; (c) when $\widehat{\delta} > \lambda_1 v_1 + \lambda_2 v_2$, we have $\delta^\dagger = \lambda_1 v_1 + \lambda_2 v_2$ and $p^\dagger = 1$, and thus no information revealing is optimal.

Proof of Lemma E.8. We prove the optimality of the signaling scheme (E.6) by constructing a feasible dual solution to the dual program $\mathcal{P}_{\texttt{OPT-Dual}}$ that satisfies the complementary slackness.

Based on the signaling scheme (E.6), we give our dual solution to the program $\mathcal{P}_{\texttt{OPT-Dual}}$ as follows:

$$\begin{aligned}
&\delta \in (\infty, v_1] : \alpha(\delta) = \max_{i\in[2]} - \frac{u_i \cdot (W(\delta) - W(\delta^\dagger)) + \alpha(\delta^\dagger)\cdot(\delta^\dagger - v_i)}{\delta - v_i} \\
&\delta \in (v_1, v_2] : \alpha(\delta) = \frac{-u_2(W(\delta) - W(\delta^\dagger)) + \max\left\{\frac{-u_2(W(v_2)-W(\delta^\dagger))}{v_2-\delta^\dagger}, -(\lambda_1 u_1 + \lambda_2 u_2)W'(\delta^\dagger)\right\}(\delta^\dagger - v_2)}{\delta - v_2} \\
&\delta \in (v_2, \infty] : \alpha(\delta) = 0 \\
&\qquad i = 1 : \eta(1) = \lambda_1 u_1 \cdot \left(W(\delta^\dagger) + \alpha(\delta^\dagger)\cdot\frac{\delta^\dagger - v_1}{u_1}\right) \\
&\qquad i = 2 : \eta(2) = \lambda_2 u_2 \cdot \left(W(\delta^\dagger) + \alpha(\delta^\dagger)\cdot\frac{\delta^\dagger - v_2}{u_2}\right);
\end{aligned} \tag{E.7}$$

Given the above constructed dual assignment, we first argue that when No information revealing is optimal, namely, $p^\dagger = 1$, we have $\alpha(\delta^\dagger) = -(\lambda_1 u_1 + \lambda_2 u_2)\cdot W'(\delta^\dagger)$ where $\delta^\dagger = \lambda_1 v_2 + \lambda_2 v_2$, otherwise we have $\alpha(\delta^\dagger) = -\frac{u_2\cdot(W(\delta^\dagger)-W(v_2))}{\delta^\dagger - v_2}$ where $\delta^\dagger \in [v_1, \lambda_1 v_2 + \lambda_2 v_2)$. For notation simplicity, let $\delta^{\mathsf{avg}} \triangleq \lambda_1 v_2 + \lambda_2 v_2$. To see this, note that when $p^\dagger = 1$, it must be the case $\gamma(\delta^\dagger) = \gamma(\delta^{\mathsf{avg}}) \ge \frac{u_1}{u_2}$. Recall that

$$\begin{aligned}
\gamma(\delta^{\mathsf{avg}}) &= \frac{v_1 - \delta^{\mathsf{avg}}}{v_2 - \delta^{\mathsf{avg}}} + \frac{W(v_2) - W(\delta^{\mathsf{avg}})}{v_2 - \delta^{\mathsf{avg}}}\cdot\frac{1}{W'(\delta^{\mathsf{avg}})}\cdot\left(1 - \frac{v_1 - \delta^{\mathsf{avg}}}{v_2 - \delta^{\mathsf{avg}}}\right) \\
&= -\frac{\lambda_2}{\lambda_1} + \frac{W(v_2) - W(\delta^{\mathsf{avg}})}{v_2 - \delta^{\mathsf{avg}}}\cdot\frac{1}{W'(\delta^{\mathsf{avg}})}\cdot\left(1 + \frac{\lambda_2}{\lambda_1}\right)
\end{aligned}$$

Hence, we have

$$-\frac{\lambda_2}{\lambda_1} + \frac{W(v_2) - W(\delta^{\mathsf{avg}})}{v_2 - \delta^{\mathsf{avg}}}\cdot\frac{1}{W'(\delta^{\mathsf{avg}})}\cdot\left(1 + \frac{\lambda_2}{\lambda_1}\right) \ge \frac{u_1}{u_2}\,.$$

Rearranging the above inequality gives us

$$-(\lambda_1 u_1 + \lambda_2 u_2)\cdot W'(\delta^{\mathsf{avg}}) \ge -\frac{u_2\cdot(W(\delta^{\mathsf{avg}}) - W(v_2))}{\delta^{\mathsf{avg}} - v_2}\,,$$

which implies the dual assignment of $\alpha(\delta^{\mathsf{avg}})$ when No information revealing is optimal. As a consequence, we have $\eta(2) = \lambda_2 u_2 W(v_2)$ when No information revealing is not optimal, and $\eta(2) = \lambda_2 u_2 \left(W(\delta^{\mathsf{avg}}) - (\lambda_1 u_1 + \lambda_2 u_2)\cdot W'(\delta^{\mathsf{avg}})\cdot\frac{\delta^{\mathsf{avg}} - v_2}{u_2}\right)$ when No information revealing is optimal.

In below, we show that the above constructed dual solution (E.7) is indeed a feasible solution to the dual program $\mathcal{P}_{\texttt{OPT-Dual}}$ (i.e., the following constraint (E.9) holds), and also, complementary slackness holds between (E.6) and (E.7) (i.e., the following constraint (E.8) holds).

$$W(\delta) + \alpha(\delta)\cdot\frac{\delta - v_i}{u_i} = \frac{\eta(i)}{\lambda_i u_i}, \quad \text{if } \pi_i^*(\delta) > 0,\ \forall i \in [2], \qquad \text{(complementary-slackness)} \tag{E.8}$$

$$W(\delta) + \alpha(\delta)\cdot\frac{\delta - v_i}{u_i} \le \frac{\eta(i)}{\lambda_i u_i}, \quad \text{if } \pi_i^*(\delta) = 0,\ \forall i \in [2], \qquad \text{(dual-feasibility)} \tag{E.9}$$

Complementary slackness. We now argue the complementary slackness of the constructed assignment. Namely, for each state $i \in [2]$ and $\delta \in (-\infty, \infty)$ such that $\pi_i^*(\delta) > 0$, its corresponding dual constraint holds with equality, i.e., the above equality (E.8). We verify this for each state $i \in [2]$ separately.

- Fix state 1, note that $\pi_1^*(\delta) > 0$ for $\delta = \delta^\dagger$ only. Here equality (E.8) holds by construction.

- Fix state 2, note that $\pi_2^*(\delta) > 0$ for $\delta = \delta^\dagger$ and $\delta = v_2$ only. When $\delta = \delta^\dagger$, the equality (E.8) holds for $\pi_2^*(\delta^\dagger) > 0$. To see this, when No information revealing is not optimal, we have

$$W(\delta^\dagger) + \alpha(\delta^\dagger)\cdot\frac{\delta^\dagger - u_2}{u_2} \overset{(a)}{=} W(\delta^\dagger) - \frac{u_2\cdot(W(v_2) - W(\delta^\dagger))}{v_2 - \delta^\dagger}\cdot\frac{\delta^\dagger - v_2}{u_2} = W(v_2) \overset{(b)}{=} \frac{\eta(2)}{\lambda_2 u_2}\ ,$$

where the equality (a) holds due to the assignment $\alpha(\delta^\dagger)$, and the equality (b) holds due to the assignment $\eta(2)$. When No information revealing is optimal, we have

$$W(\delta^{\mathsf{avg}}) + \alpha(\delta^{\mathsf{avg}})\cdot\frac{\delta^{\mathsf{avg}} - u_2}{u_2} \overset{(a)}{=} \frac{\eta(2)}{\lambda_2 u_2}\ ,$$

where the equality (a) directly follows from the assignment of $\eta(2)$. Now it is remaining verify equality (E.8) for $\pi_2^*(v_2)$. To see this, note that this must be the case where $\eta(2) = \lambda_2 u_2 W(v_2)$:

$$W(v_2) + \alpha(v_2)\cdot\frac{v_2 - u_2}{u_2} \overset{(a)}{=} W(v_2) \overset{(b)}{=} \frac{\eta(2)}{\lambda_2 u_2}\ ,$$

where the equalities (a) (b) hold due to the assignment $\alpha(\delta^\dagger)$ and the assignment $\eta(2)$.

Dual feasibility. when No information revealing is not optimal. Note that

$$\frac{\eta(1)}{\lambda_1 u_1} \overset{(a)}{=} W(\delta^\dagger) + \alpha(\delta^\dagger)\cdot\frac{\delta^\dagger - v_1}{u_1},\quad \frac{\eta(2)}{\lambda_2 u_2} = W(\delta^\dagger) + \alpha(\delta^\dagger)\cdot\frac{\delta^\dagger - v_2}{u_2}\ .$$

Thus, we can rewrite those dual constraints associated with $\pi_1^*(\delta)$ for state 1 and those dual constraints associated with $\pi_2^*(\delta)$ for state 2 as follows

$$W(\delta) + \alpha(\delta)\cdot\frac{\delta - v_1}{u_1} \le W(\delta^\dagger) + \alpha(\delta^\dagger)\cdot\frac{\delta^\dagger - v_1}{u_1}; \tag{E.10}$$

$$W(\delta) + \alpha(\delta)\cdot\frac{\delta - v_2}{u_2} \le W(\delta^\dagger) + \alpha(\delta^\dagger)\cdot\frac{\delta^\dagger - v_2}{u_2}\ . \tag{E.11}$$

We verify the above inequalities for different values of δ in three cases separately.

- Fix an arbitrary $\delta \in (-\infty, v_1]$. Note by the construction of the dual assignment in (E.7), we have

$$\alpha(\delta) \ge -\frac{u_1(W(\delta) - W(\delta^\dagger))}{\delta - v_1} + \alpha(\delta^\dagger)\frac{\delta^\dagger - v_1}{\delta - v_1},\quad \alpha(\delta) \ge -\frac{u_2(W(\delta) - W(\delta^\dagger))}{\delta - v_2} + \alpha(\delta^\dagger)\frac{\delta^\dagger - v_2}{\delta - v_2}$$

after rearranging the terms, the above inequalities directly imply the inequality (E.10) and and the inequality (E.11).

- Fix an arbitrary $\delta \in (v_1, v_2)$.

In this case, we first argue the feasibility of dual assignment (E.6) when No information revealing is not optimal. By construction, we have $\alpha(\delta) = -\frac{u_2\cdot(W(\delta) - W(v_2))}{\delta - v_2}$, which directly implies the inequality (E.11). To ensure the inequality (E.10), it is remaining to show that the following holds for all $\delta \in (v_1, v_2)$

$$-\frac{u_2\cdot(W(\delta) - W(v_2))}{\delta - v_2} \le \frac{-u_1\cdot(W(\delta) - W(\delta^\dagger)) + \alpha(\delta^\dagger)\cdot(\delta^\dagger - v_1)}{\delta - v_1}\ ,$$

which is equivalent to show the following holds for all $\delta \in (v_1, v_2)$

$$\frac{u_2}{u_1}\cdot\left(-\frac{W(\delta) - W(v_2)}{\delta - v_2} + \frac{W(\delta^\dagger) - W(v_2)}{\delta^\dagger - v_2}\cdot\frac{\delta^\dagger - v_1}{\delta - v_1}\right) \le \frac{W(\delta^\dagger) - W(\delta)}{\delta - v_1}\ . \tag{E.12}$$

We define following function $f(\delta) \triangleq \left(-\frac{W(\delta)-W(v_2)}{\delta-v_2} + \frac{W(\delta^\dagger)-W(v_2)}{\delta^\dagger-v_2}\cdot\frac{\delta^\dagger-v_1}{\delta-v_1}\right)\cdot\frac{\delta-v_1}{W(\delta^\dagger)-W(\delta)}$. Then the inequality (E.12) is equivalent to show that

$$\forall\delta\in(v_1,\delta^\dagger], f(\delta)\geq\frac{u_1}{u_2};\text{ and }\forall\delta\in(\delta^\dagger,v_2), f(\delta)\leq\frac{u_1}{u_2}\ . \tag{E.13}$$

Inspecting the first-order derivative of the function $f(\cdot)$, we know that $f'(\delta)<0,\forall\delta\in(v_1,v_2)$. Moreover, observe that

$$\lim_{\delta\to(\delta^\dagger)^+} f(\delta) = \frac{v_1-\delta^\dagger}{v_2-\delta^\dagger} + \frac{W(v_2)-W(\delta^\dagger)}{v_2-\delta^\dagger}\cdot\frac{1}{W'(\delta^\dagger)}\cdot\left(1-\frac{v_1-\delta^\dagger}{v_2-\delta^\dagger}\right) \overset{(a)}{=} \gamma(\delta^\dagger)\ ,$$

where the equality (a) follows from the definition of the function $\gamma(\cdot)$. Recall that by definition, when $\delta^\dagger\in(v_1,\lambda_1v_1+\lambda_2v_2)$, we must have $\gamma(\delta^\dagger)=\frac{u_1}{u_2}$, which proves the inequality (E.13). When $\delta^\dagger=v_1$, it suffices to argue that $\forall\delta\in(v_1,v_2), f(\delta)\leq\frac{u_1}{u_2}$. To see this, note that for all $\delta\in(v_1,v_2)$, we have $f(\delta)\leq f(\delta_1)=\lim_{\delta\to(\delta^\dagger)^+}f(\delta)=\gamma(\delta^\dagger)\leq\gamma(\widehat{\delta})=\frac{u_1}{u_2}$, where we have used the definition of $\widehat{\delta}$, and the function $\gamma(\cdot)$ is decreasing.

We now argue the feasibility of dual assignment (E.6) when No information revealing is optimal. Note that to ensure that the inequalities (E.10) and and (E.11) hold for dual assignment (E.6), it is remaining to show that

$$\frac{u_2(W(\delta^{\mathsf{avg}})-W(\delta))+\alpha(\delta^{\mathsf{avg}})(\delta^{\mathsf{avg}}-v_2)}{\delta-v_2}\leq\frac{u_1(W(\delta^{\mathsf{avg}})-W(\delta))+\alpha(\delta^{\mathsf{avg}})(\delta^{\mathsf{avg}}-v_1)}{\delta-v_1}$$

Rearranging the terms, it suffices to show that

$$\alpha(\delta^{\mathsf{avg}})\geq\frac{u_2(\delta-v_1)-u_1(\delta-v_2)}{(v_2-v_1)(\delta-\delta^{\mathsf{avg}})}\cdot(W(\delta^{\mathsf{avg}})-W(\delta)),\ \forall\delta\in(v_1,\delta^{\mathsf{avg}}];$$
$$\alpha(\delta^{\mathsf{avg}})\leq\frac{u_2(\delta-v_1)-u_1(\delta-v_2)}{(v_2-v_1)(\delta-\delta^{\mathsf{avg}})}\cdot(W(\delta^{\mathsf{avg}})-W(\delta)),\ \forall\delta\in(\delta^{\mathsf{avg}},v_2]\ .$$

Consider the function $f(\delta)\triangleq\frac{u_2(\delta-v_1)-u_1(\delta-v_2)}{(v_2-v_1)(\delta-\delta^{\mathsf{avg}})}\cdot(W(\delta^{\mathsf{avg}})-W(\delta))$, then we have $\lim_{\delta\to\delta^{\mathsf{avg}}}f(\delta) = -(\lambda_1u_1+\lambda_2u_2)\cdot W'(\delta^{\mathsf{avg}})=\alpha(\delta^{\mathsf{avg}})$. Furthermore, it can be shown that $f(\delta)\leq\alpha(\delta^{\mathsf{avg}}),\forall\delta\in(v_1,\delta^{\mathsf{avg}}]$, and $f(\delta)\geq\alpha(\delta^{\mathsf{avg}}),\forall\delta\in(\delta^{\mathsf{avg}},v_2]$.

- Fix an arbitrary $\delta\geq v_2$. Then by construction, we have

$$\alpha(\delta)=0\leq\frac{u_1(W(\delta^\dagger)-W(\delta))+\alpha(\delta^\dagger)(\delta^\dagger-v_1)}{\delta-v_1};\ \alpha(\delta)=0\leq\frac{u_2(W(\delta^\dagger)-W(\delta))+\alpha(\delta^\dagger)(\delta^\dagger-v_2)}{\delta-v_2}\ ,$$

which directly imply the inequalities (E.10) and (E.11).

□

F Omitted Proofs in Section 5

F.1 Omitted Proof of Proposition 5.1

Proof. Consider following problem instance with binary state (i.e., $m=2$),

$$\lambda_1=1-\frac{\varepsilon}{4-\varepsilon},\qquad\lambda_2=\frac{\varepsilon}{4-\varepsilon},\qquad v_1=\log\left(\frac{\varepsilon}{4-\varepsilon}\right),\qquad v_2=-\frac{\lambda_1}{\lambda_2}v_1.$$

In this problem instance, the optimal censorship $\hat{\pi}^*$ for a fully rational receiver is the no-information revealing signaling scheme, in which both states are pooled at $\hat{i}^\dagger=0$. Now, consider a receiver with bounded rationality level $\beta=1$. Note that

$$\mathbf{Payoff}_1[\hat{\pi}^*]=W(0)=\frac{1}{2}\ .$$

On the other hand, the optimal expected sender utility $\mathbf{Payoff}_1[\texttt{OPT}(1)]$ can be lower bounded by the full-information revealing signaling scheme, i.e.,

$$\mathbf{Payoff}_1[\texttt{OPT}(1)] \geq \lambda_1 W(v_1) + \lambda_2 W(v_2) > \lambda_1 W(v_1) = \left(1 - \frac{\varepsilon}{4-\varepsilon}\right)\frac{1}{1+\frac{\varepsilon}{4-\varepsilon}} = 1 - \frac{\varepsilon}{2}$$

which completes the proof. $\square$

F.2 Omitted Proof of Proposition 5.2

Proof. Given a sufficiently small $\varepsilon > 0$, consider following problem instance with three states (i.e., $m = 3$),

$$\lambda_1 = \varepsilon, \qquad \lambda_2 = \lambda_3 = \frac{1-\varepsilon}{2}, \qquad v_1 = -0.01, \qquad v_2 = 0.01, \qquad v_3 = 3\ .$$

In this problem instance, the optimal direct signaling scheme $\widetilde{\pi}^*$ for a fully rational receiver is characterized as follows

$$\begin{aligned}
&\widetilde{\pi}^*_1(\delta) = \mathbb{1}[\delta = 0] \\
&\widetilde{\pi}^*_2(\delta) = \frac{2\varepsilon}{1-\varepsilon}\mathbb{1}[\delta = 0]\,, \quad \widetilde{\pi}^*_2(\delta) = \left(1 - \frac{2\varepsilon}{1-\varepsilon}\right)\mathbb{1}\left[\delta = \frac{\frac{0.01(1-3\varepsilon)}{2} + \frac{3(1-\varepsilon)}{2}}{1-2\varepsilon}\right] \\
&\widetilde{\pi}^*_3(\delta) = \mathbb{1}\left[\delta = \frac{\frac{0.01(1-3\varepsilon)}{2} + \frac{3(1-\varepsilon)}{2}}{1-2\varepsilon}\right]\ .
\end{aligned}$$

On the other hand, for a sufficiently small ε, it can be shown that from Theorem 3.1, the optimal signaling scheme $\texttt{OPT}(\beta)$ for any bounded rationality level $\beta < \infty$ is full-information revealing, yielding $\mathbf{Payoff}_\beta[\texttt{OPT}(\beta)] = \varepsilon W(-0.01) + W(0.01)(1-\varepsilon)/2 + W(3)(1-\varepsilon)/2$. Numerically, it can be verified that $\lim_{\beta\to\infty}\lim_{\varepsilon\to 0}\frac{\mathbf{Payoff}_\beta[\texttt{OPT}(\beta)]}{\mathbf{Payoff}_\beta[\widetilde{\pi}^*]} = \infty$, which completes the proof. $\square$

F.3 Omitted Proof of Theorem 5.2

THEOREM 5.2. *In SDSU environments, there exists a problem instance (Example 5.2) with binary state such that for any signaling scheme π and any $\beta_0 \geq 0$, the rationality-robust approximation ratio with respect to $\mathcal{B} = [\beta_0, \infty)$ is unbounded, i.e., $\Gamma(\pi, [\beta_0, \infty)) = \infty$.*

Proof of Theorem 5.2. For the ease of the presentation, let $\Gamma' \triangleq \frac{1}{\Gamma}$. To analyze the rationality-robust approximation ratio Γ of a certain signaling scheme π over all possible $\beta \in \mathcal{B}$, we consider following factor-revealing program

$$(\mathcal{P}_{\texttt{Factor-Revealing}}) \qquad \begin{array}{lll}
\max\limits_{\boldsymbol{\pi}\geq\mathbf{0},\Gamma'\geq 0} & \Gamma' & \text{s.t.} \\
& \lambda_1\pi_1(\delta)\cdot(\delta - v_1) + \lambda_2\pi_2(\delta)\cdot(\delta - v_2) \geq 0 & \delta \in (-\infty,\infty) \\
& \displaystyle\int_{-\infty}^{\infty} \pi_i(\delta)d\delta = 1 & i \in [2] \\
& \pi_i(\delta) \geq 0 & \delta \in (-\infty,\infty),\ i \in [2] \\
& \mathbf{Payoff}_{\beta_\ell}[\pi] \geq \Gamma'\mathbf{Payoff}_{\beta_\ell}[\texttt{OPT}(\beta_\ell)]\,, & \ell \in [L]
\end{array}$$

We first lower bound the optimal expected sender utility under the bounded rationality level β_ℓ: $\mathbf{Payoff}_{\beta_\ell}[\texttt{OPT}(\beta_\ell)] = \Omega\left(\frac{1}{\beta_\ell\cdot\exp(\beta_\ell)}\right)$. To see this, consider following signaling scheme π':

$$\pi'_1\left(\frac{\beta_\ell+2}{\beta_\ell+1}\right) = 1; \quad \pi'_2\left(\frac{\beta_\ell+2}{\beta_\ell+1}\right) = \frac{1}{\beta_\ell},\ \pi'_2(2) = 1 - \frac{1}{\beta_\ell}\ .$$

Clearly, the above signaling scheme is a feasible solution to the program $\mathcal{P}_{\texttt{OPT-Primal}}$ with the above constructed problem instance $\mathcal{I}$. Thus, we have

$$\mathbf{Payoff}_{\beta_\ell}[\texttt{OPT}(\beta_\ell)] \geq \mathbf{Payoff}_{\beta_\ell}[\pi'] \geq \lambda_2\cdot\frac{1}{\beta_\ell}W\left(\frac{\beta_\ell+2}{\beta_\ell+1}\right) = \Omega\left(\frac{1}{\beta_\ell\exp(\beta_\ell)}\right)\ .$$

Recall that $u_1 = 0$, and thus $\mathbf{Payoff}_{\beta_\ell}[\pi] = \lambda_2 u_2 \int_{-\infty}^{\infty} \pi_i(\delta) W^{(\beta_\ell)}(\delta) d\delta$. To analyze the objective of the program $\mathcal{P}_{\texttt{Factor-Revealing}}$, we consider following dual program $\mathcal{P}_{\texttt{FR-Dual}}$ of the program $\mathcal{P}_{\texttt{Factor-Revealing}}$ with relaxing its fourth constraint to $\mathbf{Payoff}_{\beta_\ell}[\pi] \geq \Gamma' \frac{1}{\beta_\ell \exp(\beta_\ell)}, \forall \ell \in [L]$:[30]

$$
(\mathcal{P}_{\texttt{FR-Dual}}) \qquad
\begin{array}{lll}
\min\limits_{\boldsymbol{\alpha},\boldsymbol{\eta},\boldsymbol{\tau}} & \eta(1) + \eta(2) & \text{s.t.} \\
& \alpha(\delta) \cdot (1 - \delta) + 2\eta(1) \geq 0 & \delta \in (-\infty, \infty) \\
& \alpha(\delta) \cdot (2 - \delta) + 2\eta(2) \geq \displaystyle\sum_{\ell=1}^{L} \tau(\ell) \cdot \frac{1}{1 + \exp(\beta_\ell \delta)} & \delta \in (-\infty, \infty) \\
& \displaystyle\sum_{\ell=1}^{L} \tau(\ell) \cdot \frac{1}{\beta_\ell \exp(\beta_\ell)} \geq 1 & \\
& \alpha(\delta) \geq 0, & \delta \in (-\infty, \infty) \\
& \tau(\ell) \geq 0 & \ell \in [L]
\end{array}
$$

Below we construct an assignment for the dual variables $\eta(1), \eta(2)$ and $\{\tau(\ell)\}$ for the dual program $\mathcal{P}_{\texttt{FR-Dual}}$ and show that together with an assignment of $\{\alpha(\delta)\}$, our constructed assignment is feasible for sufficiently large L. Consider the following dual assignment of $\eta(1), \eta(2)$ and $\{\tau(\ell)\}$,

$$
\eta(1) \leftarrow \frac{3}{L} \quad \eta(2) \leftarrow \frac{2}{L}; \qquad \tau(\ell) \leftarrow \frac{1}{L}(\beta_\ell \exp(\beta_\ell)) \quad \forall \ell \in [L]
$$

Note that the dual constraint for primal variable Γ is satisfied by construction. Next, we discuss how to construct the assignment for $\{\alpha(\delta)\}$. We consider the three cases separately: $\delta \leq 1$, $\delta \geq 2$ and $\delta \in (1, 2)$.

- For every $\delta \leq 1$, let $\alpha(\delta) = \infty$ is sufficient to satisfies the dual constraints for $\pi_1(\delta)$ and $\pi_2(\delta)$.
- For every $\delta \geq 2$, let $\alpha(\delta) = 0$ is sufficient to satisfies the dual constraints for $\pi_1(\delta)$ and $\pi_2(\delta)$. To see this, note that the dual constraints for $\pi_1(\delta)$ holds straightforwardly as $\eta(1) = {}^{3}\!/\!{}_{L} \geq 0$. To satisfy the dual constraints for $\pi_2(\delta)$, it is sufficient to show $\eta(2) \geq \frac{1}{2} \cdot \sum_{\ell=1}^{L} \frac{\tau(\ell)}{1+\exp(\beta_\ell \delta)}$, which holds for sufficiently large L. To see this, note that

$$
\begin{aligned}
\frac{1}{2} \cdot \sum_{\ell=1}^{L} \frac{\tau(\ell)}{1 + \exp(\beta_\ell \delta)} = \frac{1}{2} \cdot \sum_{\ell=1}^{L} \frac{1}{L} \frac{\beta_\ell \exp(\beta_\ell)}{1 + \exp(\beta_\ell \delta)} &\overset{(a)}{\leq} \frac{1}{2} \cdot \sum_{\ell=1}^{L} \frac{1}{L} \frac{\beta_\ell \exp(\beta_\ell)}{1 + \exp(2\beta_\ell)} \overset{(b)}{\leq} \frac{1}{2} \cdot \sum_{\ell=1}^{L} \frac{1}{L} \frac{1}{L} \\
&= \frac{1}{2L} \leq \eta(2)
\end{aligned}
$$

- Now we consider $\delta \in (1, 2)$. To satisfies the dual constraints for $\pi_1(\delta)$ and $\pi_2(\delta)$, it is sufficient to show $\frac{2\eta(1)}{\delta - 1} \geq 0$, which holds straightforwardly as $\delta > 1$ as $\eta(1) = {}^{3}\!/\!{}_{L} > 0$, and

$$
\frac{1}{2 - \delta} \cdot \left(\sum_{\ell=1}^{L} \tau(\ell) \cdot \frac{1}{1 + \exp(\beta_\ell \delta)} - 2\eta(2) \right) \leq \frac{2\eta(1)}{\delta - 1} .
$$

  Rearranging the above inequality, we have

$$
\eta(1) + \frac{\delta - 1}{2 - \delta} \cdot \eta(2) \geq \frac{1}{2} \cdot \frac{\delta - 1}{2 - \delta} \cdot \sum_{\ell=1}^{L} \frac{\tau(\ell)}{1 + \exp(\beta_\ell \delta)} . \tag{F.14}
$$

  To see why the above inequality (F.14) holds true, we consider two subcases $\delta \in [{}^{3}\!/\!{}_{2}, 2)$ and $\delta \in (1, {}^{3}\!/\!{}_{2}]$ separately.

[30] The the duality of our infinite-dimensional LP can be obtained formally from Theorem 3.12 in Anderson and Nash (1987).

Fix an arbitrary $\delta \in [3/2, 2)$. Note that for sufficiently large L

$$\text{Left-hand side of (F.14)} \ge \frac{\delta-1}{2-\delta}\eta(2) \ge \frac{1}{2-\delta}\frac{1}{L}$$

$$\begin{aligned}\text{Right-hand side of (F.14)} &\overset{(a)}{\le} \frac{1}{2}\frac{1}{2-\delta}\sum_{\ell=1}^{L}\frac{1}{L}\frac{\beta_\ell \exp(\beta_\ell)}{1+\exp(\frac{3}{2}\beta_\ell)} \\ &\overset{(b)}{\le} \frac{1}{2}\frac{1}{2-\delta}\sum_{\ell=1}^{L}\frac{1}{L}\frac{1}{L} = \frac{1}{2-\delta}\frac{1}{2L}\end{aligned}$$

where inequality (a) holds since $\delta \in [3/2, 2)$; and inequality (b) holds since $\frac{\beta_\ell \exp(\beta_\ell)}{1+\exp(\frac{3}{2}\beta_\ell)} \le \frac{1}{L}$ for sufficiently large L.

Fix an arbitrary $\delta \in (1, 3/2]$. Let $k \in \mathbb{N}$ be the index such that $\delta \in \left[1+\frac{1}{L^{k+\frac{1}{2}}}, 1+\frac{1}{L^{k-\frac{1}{2}}}\right]$. Note that for sufficiently large L,

$$\text{Left-hand side of (F.14)} \ge \eta(1) = \frac{3}{L}$$

$$\begin{aligned}\text{Right-hand side of (F.14)} &\overset{(a)}{\le} (\delta-1)\sum_{\ell=1}^{L}\frac{1}{L}\frac{\beta_\ell\exp(\beta_\ell)}{1+\exp(\delta\beta_\ell)} \\ &= (\delta-1)\sum_{\ell\in[L]:\ell\ne k}\frac{1}{L}\frac{\beta_\ell\exp(\beta_\ell)}{1+\exp(\delta\beta_\ell)} + (\delta-1)\frac{1}{L}\frac{L^k\exp(L^k)}{1+\exp(\delta L^k)} \\ &\overset{(b)}{\le} \sum_{\ell\in[L]:\ell\ne k}\frac{1}{L}\frac{1}{L} + \frac{1}{e}\frac{1}{L} \le \frac{2}{L}\end{aligned}$$

where inequality (a) holds since $\delta \in (1, 3/2]$, and inequality (b) holds since $(\delta-1)\frac{\beta_\ell\exp(\beta_\ell)}{1+\exp(\delta\beta_\ell)} \le \frac{1}{L}$ for every $\ell \ne k$ when L is sufficiently large, and $(\delta-1)\frac{L^k\exp(L^k)}{1+\exp(\delta L^k)} \le \frac{1}{e}$.

Hence, we have the optimal objective value of the program $\mathcal{P}_{\texttt{Factor-Revealing}}$ is at most $\eta(1)+\eta(2) = 5/L$. As a result, the rationality-robust approximation ratio $\Gamma(\pi, \mathcal{B})$ is at least $L/5$ for any signaling scheme π. The proof now completes. □

F.4 Positive Results for Binary-state Instances

PROPOSITION F.1. *In SDSU environments, for any problem instance $\mathcal{I} = (m, \{\lambda_i\}, \{v_i\}, \{u_i\})$ with binary state (i.e., $m = 2$), for any $K \ge 1$ and $\beta_0 \ge \frac{\lambda_2}{\lambda_1}\cdot\frac{1}{v_2-\max\{v_1,0\}}\cdot\mathbb{1}[v_2>0]$, there exists a signaling scheme π such that $\Gamma(\pi, [\beta_0, K\beta_0]) \le \left(4\sqrt{eK}+1\right)^2$.*

Here we sketch the high-level idea for the proof of Proposition F.1. Note that by definition, a rationality-robust signaling scheme with rationality-robust approximation ratio Γ must imply that it is also Γ-approximately optimal to the optimal signaling scheme $\texttt{OPT}(\beta)$ under every possible rationality level $\beta \in \mathcal{B}$. Hence, to identify a robust signaling scheme, ideally, one needs to understand how does the optimal sender expected utility (or a non-trivial upper bound of it) change when receiver's bounded rationality level changes. However, it is difficult to characterize the optimal sender expected utility[31], let alone to understand its behavior over different rationality levels. We tackle this challenge by first showing that a censorship signaling scheme whose pooling signal (i.e., roughly at $v_1 + \Theta(1/\beta)$) depends on the value of receiver's rationality level is approximately optimal to the optimal signaling scheme $\texttt{OPT}(\beta)$. With this structure of censorship signaling scheme, a robust signaling scheme can be constructed by fine-tuning the location of the pooling signal.

[31]Although in Lemma E.5, we show that optimal signaling scheme $\texttt{OPT}(\beta)$ for binary-state instances in SDSU environments is a censorship signaling scheme, the pooling signal no longer admits a simple structure as in SISU environments (see Lemma E.8). Moreover, unlike in SISU environments where the pooling signal is monotone with respect to the rationality level, here, the pooling signal of optimal signaling scheme $\texttt{OPT}(\beta)$ is no longer monotone.

LEMMA F.1. *In SDSU environments, for any problem instance $\mathcal{I} = (m, \{\lambda_i\}, \{v_i\}, \{u_i\})$ with binary state (i.e., $m = 2$), for any receiver with bounded rationality level β, if $\beta \geq \frac{\lambda_2}{\lambda_1} \cdot \frac{1}{v_2 - \max\{v_1, 0\}} \cdot \mathbb{1}[v_2 > 0]$, for any $K \geq 1$, the optimal expected sender utility* $\textbf{Payoff}[\texttt{OPT}(\beta)]$ *is at most*

$$\textbf{Payoff}[\texttt{OPT}(\beta)] \leq 16eK\textbf{Payoff}\left[\pi^\dagger\right] + \textbf{Payoff}\left[\pi^\ddagger\right] \ ,$$

where signaling scheme $\pi^\dagger$ is a censorship signaling scheme with threshold state $i^\dagger = 2$ and pooling signal $\delta^\dagger = \min\{\lambda_1 v_1 + \lambda_2 v_2, \max\{v_1, 0\} + {}^1/(K\beta)\}$, and signaling scheme $\pi^\ddagger$ is the full-information revealing signaling scheme (i.e., also a censorship signaling scheme).

Proposition F.1 is a direct implication of Lemma F.1, whose proof is deferred to Appendix F.5.

Proof of Proposition F.1. Let $\pi^\dagger$ be the signaling scheme with threshold state $i^\dagger = 2$ and pooling signal $\delta^\dagger = \min\{\lambda_1 v_1 + \lambda_2 v_2, \max\{v_1, 0\} + {}^1/(K\beta_0)\}$, and $\pi^\ddagger$ be the full-information revealing signaling scheme. Now construct signaling scheme $\pi \triangleq q\pi^\dagger + (1-q)\pi^\ddagger$ as the convex combination of signaling scheme $\pi^\dagger$ and $\pi^\ddagger$. We specify the convex combination factor $q \in (0, 1)$ in the end of the analysis. By construction, for any bounded rationality level β, $\textbf{Payoff}_\beta[\pi] = q\textbf{Payoff}_\beta\left[\pi^\dagger\right] + (1-q)\textbf{Payoff}_\beta\left[\pi^\ddagger\right]$. To see the rationality-robustness of signaling scheme π, consider a receiver with an arbitrary bounded rationality level $\beta \in [\beta_0, K\beta_0]$, and note that

$$\begin{aligned}\textbf{Payoff}_\beta[\texttt{OPT}(\beta)] &\overset{(a)}{\leq} 16e\frac{K\beta_0}{\beta}\textbf{Payoff}_\beta\left[\pi^\dagger\right] + \textbf{Payoff}_\beta\left[\pi^\ddagger\right] \\ &\overset{(b)}{\leq} 16eK\textbf{Payoff}_\beta\left[\pi^\dagger\right] + \textbf{Payoff}_\beta\left[\pi^\ddagger\right] \leq \left(\frac{16eK}{q} + \frac{1}{1-q}\right)\textbf{Payoff}_\beta[\pi]\end{aligned}$$

where inequality (a) holds from Lemma F.1, and inequality (b) holds since ${}^{\beta_0}/{}_{\beta} \leq 1$. We finishes the proof by letting the convex combination factor q minimizes ${}^{(16eK)}/{}_q + {}^1/(1-q)$. $\square$

F.5 Omitted Proof of Lemma F.1

LEMMA F.1. *In SDSU environments, for any problem instance $\mathcal{I} = (m, \{\lambda_i\}, \{v_i\}, \{u_i\})$ with binary state (i.e., $m = 2$), for any receiver with bounded rationality level β, if $\beta \geq \frac{\lambda_2}{\lambda_1} \cdot \frac{1}{v_2 - \max\{v_1, 0\}} \cdot \mathbb{1}[v_2 > 0]$, for any $K \geq 1$, the optimal expected sender utility* $\textbf{Payoff}[\texttt{OPT}(\beta)]$ *is at most*

$$\textbf{Payoff}[\texttt{OPT}(\beta)] \leq 16eK\textbf{Payoff}\left[\pi^\dagger\right] + \textbf{Payoff}\left[\pi^\ddagger\right] \ ,$$

where signaling scheme $\pi^\dagger$ is a censorship signaling scheme with threshold state $i^\dagger = 2$ and pooling signal $\delta^\dagger = \min\{\lambda_1 v_1 + \lambda_2 v_2, \max\{v_1, 0\} + {}^1/(K\beta)\}$, and signaling scheme $\pi^\ddagger$ is the full-information revealing signaling scheme (i.e., also a censorship signaling scheme).

Proof. Since the expected sender utility generated from state 1 in the optimal signaling scheme π^* is at most $\textbf{Payoff}\left[\pi^\ddagger\right]$, it is sufficient to show that the expected sender utility generated from state 2 in the optimal signaling scheme π^* is at most $16eK\textbf{Payoff}\left[\pi^\dagger\right]$. This can be further reformulated as showing $\textbf{Payoff}[\pi^*] \leq 16eK\textbf{Payoff}\left[\pi^\dagger\right]$ for all problem instances with sender utility $u_1 = 0$. We further assume $v_2 > 0$.[32] We show this inequality using the weakly duality of linear program $\mathcal{P}_{\texttt{OPT-Primal}}$ with its dual program $\mathcal{P}_{\texttt{OPT-Dual}}$.

First, consider following assignment for dual variables $\{\eta(1), \eta(2)\}$,

$$\eta(1) = \eta(2) = 8eK\,W(\delta^\dagger)$$

where $p^\dagger = \frac{\lambda_1(\delta^\dagger - v_1)}{\lambda_2(v_2 - \delta^\dagger)}$ is the threshold state probability in signaling scheme $\pi^\dagger$.

Below we argue that there exists an assignment for dual variables $\{\alpha(\delta)\}$, which together with the constructed dual assignment for $\{\eta(1), \eta(2)\}$ above is feasible. For every $\delta \in (-\infty, \infty)$, there are two dual constraints related to $\{\alpha(\delta)\}$,

$$\lambda_1\left(v_1 - \delta\right)\alpha(\delta) + \eta(1) \geq \lambda_1 u_1 W(\delta); \quad \lambda_2\left(v_2 - \delta\right)\alpha(\delta) + \eta(2) \geq \lambda_2 u_2 W(\delta)$$

[32]When sender utility $u_1 = 0$ and receiver utility difference $v_2 \leq 0$, signaling scheme $\pi^\dagger$ becomes the no-information revealing signaling scheme, which is indeed optimal due to the concavity of function $W(\cdot)$ on $(-\infty, 0]$.

If $\delta \in (-\infty, v_1]$ (resp. $\delta \in [v_2, \infty)$), setting $\alpha(\delta) = -\infty$ (resp. $\alpha(\delta) = \infty$) satisfies the dual constraints. If $\delta \in (v_1, v_2)$, plugging the assignment for dual variables $\{\eta(1), \eta(2)\}$ constructed above as well as $u_1 = 0$, the two dual constraints are equivalent to

$$\frac{\lambda_2(v_2 - \delta)}{\lambda_1(\delta - v_1)} p^\dagger W(\delta^\dagger) + p^\dagger W(\delta^\dagger) \ge \frac{1}{8eK} W(\delta) \tag{F.15}$$

Let $\delta^{\mathsf{avg}} \triangleq (\lambda_1 v_1 + \lambda_2 v_2)$. To establish the above inequality (F.15) with the dual assignment of $\{\eta(1), \eta(2)\}$, we analyze two cases (i) $\delta^{\mathsf{avg}} > \max\{v_1, 0\} + 1/(K\beta)$ and (ii) $\delta^{\mathsf{avg}} \le \max\{v_1, 0\} + 1/(K\beta)$ separately.

Suppose $\delta^{\mathsf{avg}} > \max\{v_1, 0\} + 1/(K\beta)$, and thus $\delta^\dagger = \max\{v_1, 0\} + 1/(K\beta)$. Here we consider following three subcases based on different values of δ:

- Fix an arbitrary $\delta \in (v_1, \delta^\dagger]$. Note that

$$\frac{\lambda_2(v_2 - \delta)}{\lambda_1(\delta - v_1)} p^\dagger W(\delta^\dagger) + p^\dagger W(\delta^\dagger) \ge \frac{\lambda_2(v_2 - \delta)}{\lambda_1(\delta - v_1)} p^\dagger W(\delta^\dagger) \overset{(a)}{\ge} \frac{1}{p^\dagger} p^\dagger W(\delta^\dagger) \overset{(b)}{\ge} \frac{1}{1+e} W(\delta)$$

  where inequality (a) holds since $\lambda_2(v_2 - \delta)/\lambda_1(\delta - v_1) \ge 1/p^\dagger$ due to the construction of $p^\dagger$; and inequality (b) holds since $W(\delta^\dagger) \ge W(\delta)/(1 + \exp(K))$ due to the construction of $\delta^\dagger$ and $K \ge 1$

- Fix an arbitrary $\delta \in (\delta^\dagger, (\max\{v_1, 0\} + v_2)/2]$. Note that

$$\begin{aligned}
\frac{\lambda_2(v_2 - \delta)}{\lambda_1(\delta - v_1)} p^\dagger W(\delta^\dagger) + p^\dagger W(\delta^\dagger) \ge& \frac{\lambda_2(v_2 - \delta)}{\lambda_1(\delta - v_1)} p^\dagger W(\delta^\dagger) \\
=& \frac{\lambda_2(v_2 - \delta)}{\lambda_1(\delta - v_1)} \frac{\lambda_1(\delta^\dagger - v_1)}{\lambda_2(v_2 - \delta^\dagger)} W(\delta^\dagger) \\
=& \frac{(v_2 - \delta)}{(v_2 - \delta^\dagger)} \frac{(\delta^\dagger - v_1)}{(\delta - v_1)} W(\delta^\dagger) \\
\overset{(a)}{\ge}& \frac{1}{2} \frac{1}{2e} \frac{1}{K\beta} \frac{1}{\delta - v_1} \frac{1}{\exp(\beta v_1)} \\
=& \frac{\exp(\beta\delta)}{2\beta(\delta - v_1)\exp(\beta v_1)} \cdot \frac{1}{2eK} \frac{1}{\exp(\beta\delta)} \overset{(b)}{\ge} \frac{1}{2eK} W(\delta)
\end{aligned}$$

  where inequality (a) holds since $(v_2 - \delta)/(v_2 - \delta^\dagger) \ge 1/2$, $\delta^\dagger - v_1 \ge 1/(K\beta)$, and $2\exp(K)W(\delta^\dagger) \ge 1/\exp(\beta v_1)$. To see why inequality (b) holds, first note that $1/\exp(\beta\delta) \ge W(\delta)$. Additionally, $\beta(\delta - v_1) \ge 1$, and thus $\exp(\beta\delta)/(2\beta(\delta - v_1)\exp(\beta v_1)) \ge \exp(1)/2 \ge 1$.

- Fix an arbitrary $\delta \in ((\max\{v_1, 0\} + v_2)/2, v_2)$. Note that

$$\begin{aligned}
\frac{\frac{\lambda_2(v_2-\delta)}{\lambda_1(\delta-v_1)} p^\dagger W(\delta^\dagger) + p^\dagger W(\delta^\dagger)}{\frac{1}{8eK} W(\delta)} &\ge 8eK p^\dagger \frac{W(\delta^\dagger)}{W(\delta)} \\
&\overset{(a)}{\ge} 4eK \frac{\lambda_1(\delta^\dagger - v_1)}{\lambda_2(v_2 - \delta^\dagger)} \frac{\exp(\beta(\delta - \max\{v_1, 0\}))}{\exp(1/K)} \\
&\overset{(b)}{\ge} 4K \frac{\lambda_1 \frac{1}{K\beta}}{\lambda_2(v_2 - \max\{v_1, 0\})} (\beta(\delta - \max\{v_1, 0\}))^2 \\
&\overset{(c)}{\ge} \frac{\lambda_1}{\lambda_2} \beta(v_2 - \max\{v_1, 0\}) \overset{(d)}{\ge} 1
\end{aligned}$$

  where inequality (a) holds since $2W(\delta^\dagger)/W(\delta) \ge \exp(\beta(\delta - \max\{v_1, 0\} - 1/(K\beta)))$ and the definition of $p^\dagger$; inequality (b) holds since $\delta^\dagger - v_1 \ge 1/(K\beta)$, $v_2 - \delta^\dagger \le v_2 - \max\{v_1, 0\}$, $\exp(\beta(\delta - \max\{v_1, 0\})) \ge (\beta(\delta - \max\{v_1, 0\}))^2$; inequality (c) holds since $\delta \ge (\max\{v_1, 0\} + v_2)/2$; and inequality (d) holds since we assume that $\beta \ge \lambda_2/(\lambda_1(v_2 - \max\{v_1, 0\}))$ and $\lambda_2/(\lambda_1(v_2 - \max\{v_1, 0\})) \ge 0$ due to $v_2 > 0$.

Suppose $\delta^{\mathsf{avg}} \le \max\{v_1, 0\} + 1/(K\beta)$, and thus $\delta^\dagger = \delta^{\mathsf{avg}}$. Note that in this case, the threshold state probability $p^\dagger = 1$, and inequality (F.15) can be further simplified and relaxed as $W(\delta^\dagger) \ge W(\delta)/(1+e)$ which holds for every $\delta \in (v_1, \delta^\dagger]$ due to the same argument as the previous case; and for every $\delta \in [\delta^\dagger, v_2)$ due to the monotonicity of function $W(\cdot)$.

Finally, recall that the expected sender utility in signaling scheme $\mathbf{Payoff}[\pi^\dagger]$ is $\lambda_2 u_2(p^\dagger W(\delta^\dagger) + (1-p^\dagger)W(v_2))$, which is a $(16eK)$-approximation to the objective value of the constructed dual assignment, *i.e.*, $\eta(1) + \eta(2) = 16eK\lambda_2 u_2 p^\dagger W(\delta^\dagger)$. Invoking the weak duality of linear program finishes the proof. $\square$

G The Complexity on Computing Approximately Optimal Signaling Schemes

In this section, we discuss the complexity on computing an approximately optimal signaling scheme in both SISU and SDSU environments.

PROPOSITION G.1. *In SISU environments, there exists a* $\mathsf{poly}(m)$ *time algorithm that can find the optimal signaling scheme.*

Proof. Recall that by Theorem 3.1, the optimal signaling scheme in SISU environments is a censorship signaling scheme. Thus, to find the optimal signaling scheme in SISU environments, it suffices to find the threshold state $i^\dagger$ and the threshold state probability $p^\dagger$. To identify $i^\dagger, p^\dagger$, consider the following procedure: for every state $i \in [m]$ where $v_i \ge 0$, compute the corresponding p_i where p_i is defined as in (D.2). Then the threshold state $i^\dagger = \arg\max_{i: v_i > 0}\{p_i\}$. If $p_{i^\dagger} > 1$, then the threshold state probability $p^\dagger = 1$, otherwise $p^\dagger = p_{i^\dagger}$. It is easy to see that the above procedure has the complexity at most $O(m)$. $\square$

Unlike in SISU environments, determining the computational complexity of finding the optimal signaling scheme in SDSU environments is much more challenging. One reason for this is the lack of a clear structure for the optimal signaling scheme in SDSU environments. Nonetheless, we present two complexity characterizations for finding the optimal signaling scheme in SDSU environments. The first one applies to special instances with binary states, while the second one applies to general problem instances.

COROLLARY G.1. *In SDSU environments with binary states, there exists a* $O(1)$ *time algorithm that can find the optimal signaling scheme.*

Proof. By Lemma E.5, we know that the optimal signaling scheme in SDSU environments with binary states is also a censorship signaling scheme, then the above result immediately follows by the similar analysis of Proposition G.1. $\square$

PROPOSITION G.2. *In SDSU environments, there exists a* $\mathsf{poly}(m, \beta(v_m - v_1)/\varepsilon)$ *time algorithm that can find a* $(1+\varepsilon)$*-approximate signaling scheme.*

Proof. Given an arbitrary small $\varepsilon_0 > 0$, we discuss how to solve a $(1+\varepsilon_0)$-approximate signaling scheme with running time $\mathsf{poly}(m, \beta(v_m - v_1)/\varepsilon_0)$.

Let $\varepsilon = \varepsilon_0/\beta$. Define the set $\mathcal{S} \triangleq \{v_0, v_0+\varepsilon, v_0+2\varepsilon, \dots, v_m\} \cup \{v_i\}_{i\in[m]}$. We consider the following program and its optimal solution $\widehat{\pi}^*$:

$$
(\mathcal{P}_{\texttt{OPT-Primal}}(\varepsilon)) \qquad
\begin{aligned}
\widehat{\pi}^* = \arg\max_{\boldsymbol{\pi}\ge \mathbf{0}} \quad & \sum\nolimits_{i\in[m]} \lambda_i u_i \sum_{\delta\in\mathcal{S}} \pi_i(\delta) W(\delta) && \text{s.t.} \\
& \sum\nolimits_{i\in[m]} \lambda_i (v_i - \delta)\, \pi_i(\delta) = 0 && \delta \in \mathcal{S} \\
& \sum_{\delta\in\mathcal{S}} \pi_i(\delta) = 1 && i \in [m]
\end{aligned}
$$

Essentially, the above program restricts the support of each conditional distribution $\pi_i, i \in [m]$ to be the subset of the set $\mathcal{S}$. Recall that from Lemma 4.3, we know that there exists an optimal signaling scheme π^* such that for every $\delta \in (-\infty, \infty)$, we have $|\{i : \pi_i^*(\delta) > 0, i \in [m]\}| \le 2$. Based on the signaling scheme π^*, we below construct a new signaling scheme $\widehat{\pi}$ that is also a feasible solution to the program $\mathcal{P}_{\texttt{OPT-Primal}}(\varepsilon)$. In particular, for every δ where $|\{i : \pi_i^*(\delta) > 0, i \in [m]\}| \ge 1$:

1. if $|\{i' : \pi^*_{i'}(\delta) > 0, i' \in [m]\}| = \{i\}$, let $\widehat{\pi}_i(\delta) = \pi^*_i(\delta)$;

2. if $|\{i' : \pi^*_{i'}(\delta) > 0, i' \in [m]\}| = \{i, j\}$ where $i < j$, let $\delta_L \triangleq \max\{x \in \mathcal{S} : x \le \delta\}$, and let $\delta_R \triangleq \min\{x \in \mathcal{S} : x \ge \delta\}$. Let $\widehat{\pi}_i(\delta_L) = \frac{v_j - \delta_L}{v_j - v_i} \frac{1}{\lambda_i} \frac{\pi^*(\delta)(\delta_R - \delta)}{\delta_R - \delta_L}$ and $\widehat{\pi}_i(\delta_R) = \pi^*_i(\delta) - \widehat{\pi}_i(\delta_L)$; $\widehat{\pi}_j(\delta_L) = \frac{\delta_L - v_i}{v_j - v_i} \frac{1}{\lambda_j} \frac{\pi^*(\delta)(\delta_R - \delta)}{\delta_R - \delta_L}$ and $\widehat{\pi}_j(\delta_R) = \pi^*_j(\delta) - \widehat{\pi}_j(\delta_L)$ where $\pi^*(\delta) = \lambda_i \pi^*_i(\delta) + \lambda_j \pi^*_j(\delta)$.

By construction, it is easy to verify that the signaling scheme $\widehat{\pi}$ is a feasible solution to the program $\mathcal{P}_{\texttt{OPT-Primal}}(\varepsilon)$. Furthermore, when $|\{i' : \pi^*_{i'}(\delta) > 0, i' \in [m]\}| = \{i\}$, the expected payoff contributed from the induced δ in both signaling scheme $\widehat{\pi}, \pi^*$ equals to $\lambda_i u_i \widehat{\pi}_i(\delta) W(\delta)$; when $|\{i' : \pi^*_{i'}(\delta) > 0, i' \in [m]\}| = \{i, j\}$, the expected payoff contributed from the induced δ in both signaling scheme $\widehat{\pi}, \pi^*$ satisfy that

$$\begin{aligned} & \lambda_i u_i \widehat{\pi}_i(\delta_L) W(\delta_L) + \lambda_j u_j \widehat{\pi}_j(\delta_L) W(\delta_L) + \lambda_i u_i \widehat{\pi}_i(\delta_R) W(\delta_R) + \lambda_j u_j \widehat{\pi}_j(\delta_R) W(\delta_R) \\ \overset{(a)}{\ge} \ & \lambda_i u_i \pi^*_i(\delta) \cdot W(\delta + \varepsilon) + \lambda_j u_j \pi^*_j(\delta) \cdot W(\delta + \varepsilon) j \end{aligned}$$

where in inequality (a), we have used the fact that $\pi^*_i(\delta) = \widehat{\pi}_i(\delta_L) + \widehat{\pi}_i(\delta_R), \pi^*_j(\delta) = \widehat{\pi}_j(\delta_L) + \widehat{\pi}_j(\delta_R)$, $\delta_L \le \delta \le \delta_R$, $\delta_R \le \delta + \varepsilon$, and the fact that $W(\cdot)$ is monotone non-increasing. Summing over all δ and rearranging the terms, we know that

$$\sum_{i \in [m]} \lambda_i u_i \int_\delta \pi^*_i(\delta) W(\delta + \varepsilon) d\delta \le \mathbf{Payoff}_\beta[\widehat{\pi}]$$

Now observe that for any δ, we have $\frac{W(\delta)}{W(\delta+\varepsilon)} \le \exp(\beta\varepsilon)$. Thus, with the above inequality, we have $\mathbf{Payoff}_\beta[\pi^*] \le \exp(\beta\varepsilon)\mathbf{Payoff}_\beta[\widehat{\pi}] \le (1 + \beta\varepsilon)\mathbf{Payoff}_\beta[\widehat{\pi}] = (1 + \varepsilon_0)\mathbf{Payoff}_\beta[\widehat{\pi}]$. $\square$